Did You Mean: Recursion?

Determining Recursive and Explicit Expressions from Contexts

MATERIALS

Graphic organizers from *The Password Is . . . Operations!*

Lesson Overview

Scenarios are presented that can be represented by arithmetic and geometric sequences. Students determine the value of terms in each sequence. The term recursive formula is defined and used to generate term values. As the term number increases, it becomes more time consuming to generate the term value. This sets the stage for explicit formulas to be defined and used. Students practice using these formulas to determine the values of terms in both arithmetic and geometric sequences.

Algebra 1

Number and Algebraic Methods

(12) The student applies the mathematical process standards and algebraic methods to write, solve, analyze, and evaluate equations, relations, and functions. The student is expected to:

- (C) identify terms of arithmetic and geometric sequences when the sequences are given in function form using recursive processes.
- (D) write a formula for the n^{th} term of arithmetic and geometric sequences, given the value of several of their terms.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 5.B, 5.F, 5.G

Essential Ideas

- A recursive formula expresses each new term of a sequence based on a preceding term of the sequence.
- · An explicit formula for a sequence is a formula for calculating each term of the sequence using the term's position in the sequence.
- The explicit formula for determining the *n*th term of an arithmetic sequence is $a_n = a_1 + d(n-1)$, where n is the term number, a_1 is the first term in the sequence, a_n is the *n*th term in the sequence, and d is the common difference.
- The explicit formula for determining the nth term of a geometric sequence is $g_n = g_1 \cdot r^{(n-1)}$, where n is the term number, g_1 is the first term in the sequence, g_n is the nth term in the sequence, and *r* is the common ratio.

Lesson Structure and Pacing: 2 Days

Day 1

Engage

Getting Started: Can I Get a Formula?

A scenario is given that can be represented by an arithmetic sequence. Students complete a table of values listing each term number and the value of the first ten terms. This is an introduction to the problem situation presented in Activity 3.1.

Develop

Activity 3.1: Writing Formulas for Arithmetic Sequences

Students use two worked examples to understand recursive and explicit formulas for arithmetic sequences. They use this understanding to write recursive and explicit formulas for the sequence described by the problem situation from the Getting Started. The problem situation is then changed, and students answer questions about the new problem situation by rewriting the explicit formula.

Activity 3.2: Writing Formulas for Geometric Sequences

Students are given a new problem situation and determine that the situation can be represented by a geometric sequence. They analyze two worked examples to understand recursive and explicit formulas for geometric sequences. Students then use this understanding to write recursive and explicit formulas for the sequence described by the problem situation. The problem situation is then changed, and they answer questions about the new problem situation by rewriting the explicit formula.

Day 2

Activity 3.3: Writing Recursive and Explicit Formulas

Students use what they now know about recursive and explicit formulas for arithmetic and geometric sequences to write both types of formula for each of the sequences they studied in the previous lesson.

Demonstrate

Talk the Talk: Pros and Cons

Students write paragraphs to describe the advantages and disadvantages of using recursive and explicit formulas to determine term values of arithmetic and geometric sequences.

Facilitation Notes

In this activity, a scenario is given that can be represented by an arithmetic sequence. Students complete a table of values listing each term number and the value of the first ten terms. This is an introduction to the problem situation presented in Activity 3.1.

Have students work with a partner or in a group to complete the table of values and answer Questions 1 through 4. Share responses as class.

Questions to ask

- What is the difference between an arithmetic sequence and a geometric sequence? Can you spot the difference when you first read the scenario, or do you need to observe entries listed in an organizational table?
- Does this scenario describe an arithmetic sequence or a geometric sequence? How did you determine the sequence type?
- Why is the number of home runs not the same as the term number? Does this affect how you solve for each donation amount?
- How did you determine the donation amount if the team hits 2 home runs? 9 home runs?
- What strategy can you use to calculate the *n*th term?
- What elements of a scenario are absolutely necessary to represent a situation as an arithmetic sequence?

Summary

An arithmetic sequence can be used to model a situation by creating additional term values using the common difference. The term numbers and term values can be organized in a table.

Activity 3.1 Writing Formulas for Arithmetic Sequences



DEVELOP

Facilitation Notes

In this activity, students analyze two worked examples to understand recursive and explicit formulas for arithmetic sequences. They use this understanding to write recursive and explicit formulas for the sequence described by Rico's donations to the baseball team. The problem situation is then changed, and students answer questions about the new problem situation by rewriting the explicit formula.

Ask a student to read the definition aloud. Review the Worked Example as a class.

Have students work with a partner or in a group to answer Questions 1 and 2. Share responses as class.

Questions to ask

- What is the purpose of writing a recursive formula?
- What information is needed to write a recursive formula?
- · What do the subscripts in the formula represent?
- Why is a_{n-1} rather than a_n added to d in the recursive formula?
- If n = 5, what does each term in the recursive formula represent?
- Can you determine the 11th term in this sequence without using a recursive formula?
- Why would you rather use a recursive formula in this situation?
- Would you want to use a recursive formula to identify the 200th or 1000th term value in this sequence? Why not?

Ask a student to read the information and definition following Question 2 aloud. Review the worked example as a class.

Have students work with a partner or in a group to complete Questions 3 through 5. Share responses as class.

Differentiation strategy

To assist all students, help them connect the new terminology to words they already know. Recursive has the prefix re- and means repeating something, in this case, repeating the same operation to get the next term. Explicit means clearly, such as giving explicit directions; in this case, the explicit formula is more clear or direct.

Questions to ask

- · What elements are needed to write an explicit formula?
- Why does the explicit formula use multiplication when a common difference means addition is used in the sequence?
- Why is (n-1) rather than n multiplied by d?
- When n = 1, what is the result? Explain why this makes sense.
- How do the terms of the recursive formula relate to the terms in the explicit formula?
- What is the purpose of writing an explicit formula?
- · Can you determine the 50th term in this sequence without using an explicit formula?
- Why would you rather use an explicit formula in this situation?
- Would you want to use an explicit formula or a recursive formula to identify the 202th or 935th term value in this sequence? Why?

Summary

An arithmetic sequence can be represented using a recursive formula or an explicit formula. The explicit formula is more efficient in determining any term value without having to calculate all the terms before it.

Activity 3.2 Writing Formulas for Geometric Sequences



Facilitation Notes

In this activity, students are given a problem situation that can be represented by a geometric sequence. They analyze two worked examples to understand recursive and explicit formulas for geometric sequences. They then write and use recursive and explicit formulas for the sequence described by the problem situation.

Ask a student to read the introduction aloud.

Have students work with a partner or in a group to complete Question 1. Share responses as class.

Questions to ask

- Does this sequence have a common difference or a common ratio? How do you know?
- How did you determine the common ratio?
- · Why is the number of cell divisions not the same as the term number? Does this affect how you determine the total number of cells?
- · How did you determine the total number of cells after 2 cell divisions?
- Do you need to know the 4th term value to determine the 5th term value?
- How did you determine the total number of cells after 5 cell divisions?
- Do you need to know the 9th term value to determine the 10th term value?
- How did you determine the total number of cells after 10 cell divisions?

Ask a student to read the description of the recursive formula associated with a geometric sequence aloud. Review the worked example as a class.

Have students work with a partner or in a group to complete Question 2. Share responses as class.

Questions to ask

- Is there more than one strategy to calculate the 12th term value?
- · What is the first term?
- What is the common ratio?

Ask a student to read the description of the explicit formula associated with a geometric sequence aloud. Review the worked example as a class.

Have students work with a partner or in a group to complete Questions 3 through 5. Share responses as class.

Questions to ask

- Why does the explicit formula use exponents when a common ratio means multiplication was used in the sequence?
- How do the terms of the recursive formula relate to the terms in the explicit formula?
- · How do you know when it is better to use the recursive formula and when it is better to use the explicit formula?
- How do you know what term number to use when solving the formula? Is that always the case?
- How are the term values of a geometric series affected when r is a negative value?
- How is the scenario in Question 4 different from Question 3? How do these changes affect your formula?

Misconception

Students sometimes misunderstand the meaning of the first term value of a sequence. Sequences always start with term number 1. Based upon the phrasing of the scenario, the first term number usually represents a starting value, and the 2nd term represents the first time the operation is performed. For example, the first term is the number of cells after 0 divisions, not after one division, so the 100th term represents the number of cells after 99 divisions, not after 100 divisions. Sometimes students get this concept, but go in the reverse direction. As students solve these problems, have them explain the value they substitute in the formula and the meaning of the result. The clarification now will help later when students connect sequences and functions, and realize that the first term of a sequence is not the same as the *y*-intercept.

Summary

A geometric sequence can be represented using a recursive formula or an explicit formula. The explicit formula is more efficient to determine any term value without having to calculate all the terms before it.

Activity 3.3Writing Recursive and Explicit Formulas



Facilitation Notes

In this activity, students use what they know about recursive and explicit formulas for arithmetic and geometric sequences to write both types of formulas for each of the sequences they studied in the previous lesson.

Have students work with a partner or in a group to complete this activity. Share responses with the class.

As students work, look for

- Arithmetic sequences written two different ways when d is a negative value.
- Proper use of parentheses when *r* is a negative value.

Differentiation strategy

To extend the activity, show students how to use graphing calculators to identify a specified term. These steps show how to determine the 20th term in the sequence 3, 10, 17, 24, 31 . . . using a graphing calculator.

- Step 1: Enter the first value of the sequence, 3. Then press ENTER to register the first term. The calculator can now recall that first term.
- Step 2: From that term, add the common difference, 7. Press ENTER. The next term should be calculated. The calculator can now recall the formula as well.
- Step 3: Press ENTER and the next term should be calculated.
- Step 4: Continue pressing ENTER until you determine the nth term of the sequence you want to determine. Keep track of how many times you press ENTER so you know when you have the 20th term.

These steps show how to use a graphing calculator to generate two sequences at the same time to determine a certain term in a sequence.

- Step 1: Within a set of brackets, enter the first term number followed by a comma and then the first term value of the sequence, {1,3}. Press ENTER.
- Step 2: Provide direction to the calculator to increase the term number by 1 and the term value by the common difference. Type: {Ans(1)+1, Ans(2)+7}. Press ENTER.
- Step 3: Continue pressing ENTER until you reach the *n*th term number and value you want to determine.

Questions to ask

- Where did you get the information from the sequence to create the recursive formula?
- Did you go back to the original sequence or use the recursive formula to write the explicit formula? Why?
- What information is contained in the explicit formula that is not in the recursive formula?
- How are the recursive and explicit formulas related for an arithmetic sequence? A geometric sequence?

Summary

Recursive and explicit formulas can be used to generate arithmetic and geometric sequences.

DEMONSTRATE

Talk the Talk: Pros and Cons

Facilitation Notes

In this activity, students write paragraphs describing the advantages and disadvantages of using recursive and explicit formulas to determine term values of arithmetic and geometric sequences.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Questions to ask

- · What information is needed to create an explicit formula for an arithmetic sequence?
- What information is needed to create a recursive formula for a geometric sequence?
- · Which formula requires knowledge of the previous term and the common difference?
- · Which formula requires knowledge of the term's position in the sequence?
- Which formula is used to generate the next term and depends on knowledge of the previous term?
- Which formula is used to generate any term and depends on knowledge of the term number?

Summary

There are advantages and disadvantages to using either an explicit or recursive formula to represent an arithmetic or geometric sequence.

Did You Mean: Recursion?

Determining Recursive and Explicit Expressions from Contexts

Warm Up

The local bank has agreed to donate \$250 to the annual turkey fund to help feed families in need. In addition, for every bank customer that donates \$50, the bank will donate \$25.

- 1. A sequence describes the relationship between the number of \$50 donations and the amount of the bank's donation. Is the sequence arithmetic or geometric?
- 2. How can you calculate the 10th term based on the 9th term?
- 3. What is the 20th term?

Learning Goals

- Write recursive formulas for arithmetic and geometric sequences from contexts.
- · Write explicit expressions for arithmetic and geometric sequences from contexts.
- · Use formulas to determine unknown terms of a sequence.

Key Terms

- · recursive formula
- · explicit formula

You have learned that arithmetic and geometric sequences always describe functions. How can you write equations to represent these functions?

LESSON 3: Did You Mean: Recursion? • 1

ELL Tip

Assess students' prior knowledge of the word donate. Create a list of synonyms for the word and discuss the distinction between donating money and giving money to a friend, for example. Ask for volunteers to share examples of scenarios of money donations.

Warm Up Answers

- 1. The sequence is arithmetic because the common difference is 25.
- 2. Add 25 to the ninth term.
- 3. The 20th term is \$725.

- The sequence is arithmetic. It is arithmetic because a constant is added to each term to produce the next term.
- 2. The common difference is 18.

3.

Number of Home Runs	Term Number (n)	Donation Amount (dollars)	
0	1	125	
1	2	143	
2	3	161	
3	4	179	
4	5	197	
5	6	215	
6	7	233	
7	8	251	
8	9	269	
9	10	287	

4. To calculate the tenth term, add 18 to the ninth term.

GETTING STARTED

Can I Get a Formula?



Notice that the 1st term in this sequence is the amount Rico donates if the team hits 0 home runs. While a common ratio or a common difference can help you determine the next term in a sequence, how can they help you determine the thousandth term of a sequence? The ten-thousandth term of a sequence?

Consider the sequence represented in this situation.

Rico owns a sporting goods store. He has agreed to donate \$125 to the Centipede Valley High School baseball team for their equipment fund. In addition, he will donate \$18 for every home run the Centipedes hit during the season. The sequence shown represents the possible dollar amounts that Rico could donate for the season.

125, 143, 161, 179, . . .

Number of Home Runs	Term Number (n)	Donation Amount (dollars)
0	1	
1		
2		
3		
4		
5		
6		
7		
8		
9		

- 1. Identify the sequence type. Describe how you know.
- 2. Determine the common difference or common ratio for the sequence.
- 3. Complete the table.
- 4. Explain how you can calculate the tenth term based on the ninth term.



2 · TOPIC 2: Sequences

ELL Tip

Review the terms *common difference* and *common ratio*. Create an anchor chart with two columns using the terms as the headers for each column. Discuss the similarities and differences between the terms and fill in the anchor chart with key ideas about each term. Ask students to give examples of sequences that have a *common difference*, as well as sequences that have a *common ratio*. Ensure students' understanding of which term applies to an arithmetic sequence and which term applies to a geometric sequence.

3.1

Writing Formulas for Arithmetic Sequences



A **recursive formula** expresses each new term of a sequence based on the preceding term in the sequence. The recursive formula to determine the *n*th term of an arithmetic sequence is:

*n*th term
$$a_n = \underline{a_{n-1}} + d$$
 common difference previous

You only need to know the previous term and the common difference to use the recursive formula.

Worked Example

Consider the sequence -2, -9, -16, -23, . . .

You can use the recursive formula to determine the 5th term.

$$a_n = a_{n-1} + d$$

 $a_5 = a_{5-1} + (-7)$

The expression $a_{\rm S}$ represents the 5th term. The previous term is -23, and the common difference is -7.

$$a_5 = a_4 + (-7)$$

 $a_5 = -23 + (-7)$
 $a_5 = -30$

The 5th term of the sequence is -30.

Consider the sequence showing Rico's contribution to the Centipedes baseball team in terms of the number of home runs hit.

- Use a recursive formula to determine the 11th term in the sequence. Explain what this value means in terms of this problem situation.
- 2. Is there a way to calculate the 20th term without first calculating the 19th term? If so, describe the strategy.

LESSON 3: Did You Mean: Recursion? • 3

ELL Tip

Ask students to identify what the prefix *pre*- means in the word *preceding*. Follow up with additional examples of words with the prefix *pre*-, including pretest, preview, and precooked. Define these words and then ask students to explain why preceding means "the term before" in the context of "the preceding term in the sequence". Create a list of words beginning with the prefix *pre*- and have students add to it as they encounter additional words with this prefix in the lesson.

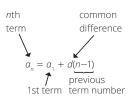
Answers

- 1. $a_{11} = a_{10} + 18$; $a_{11} = 287 + 18$; $a_{11} = 305$; Rico will donate a total of \$305 if 10 home runs are hit.
- 2. Answers will vary.

LESSON 3: Did You Mean: Recursion? • 11

You can determine the 93rd term of the sequence by calculating each term before it, and then adding 18 to the 92nd term, but this will probably take a while! A more efficient way to calculate any term of a sequence is to use an explicit formula.

An **explicit formula** of a sequence is a formula to calculate the nth term of a sequence using the term's position in the sequence. The explicit formula for determining the nth term of an arithmetic sequence is:





The 1st term in this sequence is the amount Rico donates if the team hits 0 home runs. So, the 93rd term represents the amount Rico donates if the team hits 92 home runs.

Worked Example

You can use the explicit formula to determine the 93rd term in this problem situation.

$$a_n = a_1 + d(n - 1)$$

 $a_{93} = 125 + 18(93 - 1)$

The expression $a_{\rm 93}$ represents the 93rd term. The first term is 125, and the common difference is 18.

$$a_{93} = 125 + 18(92)$$

 $a_{93} = 125 + 1656$
 $a_{93} = 1781$

The 93rd term of the sequence is 1781.

This means Rico will contribute a total of \$1781 if the Centipedes hit 92 home runs.

4 · TOPIC 2: Sequences

a. 35 home runs	b. 48 home runs	
a. 33 nome runs	D. 46 HOME FURS	
c. 86 home runs	d. 214 home runs	
	tial contribution and amount donated per ontribute \$500 and will donate \$75 for every	
4. Write the first 5 terms of contribution Rico will do	f the sequence representing the new nate to the Centipedes.	
contribution Rico will do		
contribution Rico will do	nate to the Centipedes.	
contribution Rico will do 5. Determine Rico's contrib runs hit.	nate to the Centipedes. Sution for each number of home	

3a. \$755

3b. \$989

3c. \$1673

3d. \$3977

4. 500, 575, 650, 725, 800

5a. \$3425

5b. \$4250

1a. This sequence is geometric because each term is multiplied by a constant to produce the next term. The common ratio is 2.

1b.

Number of Cell Divisions	Term Number (n)	Total Number of Cells
0	1	1
1	2	2
2	3	4
3	4	8
4	5	16
5	6	32
6	7	64
7	8	128
8	9	256
9	10	512

1c. Multiply the ninth term by 2.

3.2

Writing Formulas for Geometric Sequences



When it comes to bugs, bats, spiders, and—ugh, any other creepy crawlers—finding one in your house is finding one too many! Then again, when it comes to cells, the more the better. Animals, plants, fungi, slime, molds, and other living creatures are composed of eukaryotic cells. During growth, generally there is a cell called a "mother cell" that divides itself into two "daughter cells." Each of those daughter cells then divides into two more daughter cells, and so on.

Notice that the 1st term in this sequence is the total number of cells after 0 divisions (that is, the mother cell).

- 1. The sequence shown represents the growth of eukaryotic cells.
 - 1, 2, 4, 8, 16, . . .
 - a. Describe why this sequence is geometric and identify the common ratio.

Number of Cell Divisions	Term Number (n)	Total Number of Cells
0	1	
1		
2		
3		
4		
5		
6		
7		
8		
9		

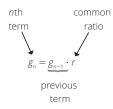
- Complete the table of values. Use the number of cell divisions to identify the term number and the total number of cells after each division.
- Explain how you can calculate the tenth term based on the ninth term.

6 · TOPIC 2: Sequences

ELL Tip

Review the scientific terms given in the example for the activity. Ask students to make a list of terms such as *cells, mother cells, daughter cells, petri dish,* and *hypothesis*. Discuss how the terms are used in the activity and ask students to create a sentence using each term to demonstrate their understanding. Also ask students to create a list of synonyms for *hypothesis*.

The recursive formula to determine the nth term of a geometric sequence is:



Worked Example

Consider the sequence shown.

You can use the recursive formula to determine the 5th term.

$$g_n = g_{n-1} \cdot r$$
$$g_5 = g_{5-1} \cdot (3)$$

The expression $g_{\rm s}$ represents the 5th term. The previous term is 108, and the common ratio is 3.

$$g_5 = g_4 \cdot (3)$$

$$g_5 = 108 \cdot (3)$$

$$g_5 = 324$$

The 5th term of the sequence is 324.

Consider the sequence of cell divisions and the total number of resulting cells.

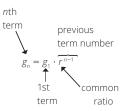
2. Write a recursive formula for the sequence and use the formula to determine the 12th term in the sequence. Explain what your result means in terms of this problem situation.

Answers

2. $g_{12} = 1024 \cdot 2$; $g_{12} = 2048$; There are a total of 2048 cells after 11 divisions.

LESSON 3: Did You Mean: Recursion? • 7

The explicit formula to determine the *n*th term of a geometric sequence is:





The 1st term in this sequence is the total number of cells after 0 divisions. So, the 20th term represents the total number of cells after 19 divisions.

Worked Example

You can use the explicit formula to determine the 20th term in this problem situation. $\;$

$$g_n = g_1 \cdot r^{n-1}$$
$$g_{20} = 1 \cdot 2^{20-1}$$

The expression g_{20} represents the 20th term. The first term is 1, and the common ratio is 2.

$$g_{20} = 1 \cdot 2^{19}$$

 $g_{20} = 1 \cdot 524,288$
 $g_{20} = 524,288$

The 20th term of the sequence is 524,288.

This means that after 19 cell divisions, there are a total of 524,288 cells.

8 · TOPIC 2: Sequences

- 3. Use the explicit formula to determine the total number of cells for each number of divisions.
 - a. 11 divisions
- b. 14 divisions
- c. 18 divisions
- d. 22 divisions

Suppose that a scientist has 5 eukaryotic cells in a petri dish. She wonders how the growth pattern would change if each mother cell divided into 3 daughter cells.

- 4. Write the first 5 terms of the sequence for the scientist's hypothesis.
- 5. Determine the total number of cells in the petri dish for each number of divisions.
 - a. 13 divisions
- b. 16 divisions

ACTIVITY

Writing Recursive and Explicit **Formulas**



In the previous lesson you identified sequences as either arithmetic or geometric and then matched a corresponding graph.

1. Go back to the graphic organizers from the previous lesson. Write the recursive and explicit formulas for each sequence.

LESSON 3: Did You Mean: Recursion? • 9

Sequence N:

$$a_n = a_{n-1} - 20.5$$

$$a_n = 1473.2 - 20.5(n - 1)$$

Sequence P:

$$g_n = g_{n-1} \cdot (-3)$$

$$g_n = -4 \cdot (-3)^{n-1}$$

Answers

1.

Sequence A:

$$g_n = g_{n-1} \cdot 2$$

$$g_n = 45 \cdot 2^{n-1}$$

Sequence B:

$$a_n = a_{n-1} + 2$$

$$a_n = -4 + 2(n-1)$$

Sequence C:

$$g_n = g_{n-1} \cdot 3$$

$$g_n = -2 \cdot 3^{n-1}$$

Sequence E:

$$a_n = a_{n-1} - \frac{9}{4}$$

$$a_n = a_{n-1} - \frac{9}{4}$$

$$a_n = 4 - \frac{9}{4}(n-1)$$

Sequence F:

$$g_n = g_{n-1} \cdot \frac{1}{10}$$

$$g_n = 1234 \cdot \left(\frac{1}{10}\right)^{n-1}$$

Sequence H:

$$a_n = a_{n-1} + 4$$

$$a_n = -20 + 4(n-1)$$

Sequence I:

$$g_n = g_{n-1} \cdot \frac{1}{2}$$

$$g_n = -5 \cdot \left(\frac{1}{2}\right)^{n-1}$$

Sequence K:

$$a_n = a_{n-1} - 1.5$$

$$a_n = 6.5 - 1.5(n - 1)$$

Sequence M:

$$g_n = g_{n-1} \cdot \left(-\frac{1}{4}\right)$$

$$g_n = g_{n-1} \cdot \left(-\frac{1}{4}\right)$$
$$g_n = -16 \cdot \left(-\frac{1}{4}\right)^{n-1}$$

1. Sample answer.

Advantage: It enables you to make sense of the growth pattern of the sequence.

Disadvantage: It is not an efficient method when determining the term value for a large term number.

2. Sample answer.

Advantage: It is an efficient method when determining the term value for a large term number.

Disadvantage: It takes a little more effort to determine an explicit formula than it does to determine a recursive formula.

