1

Gauss in Das Haus

Solving Systems of Equations

Warm Up

Use substitution to solve each system of equations.

1. $\begin{cases} 2x + 3y = 8 \\ x = -2 \end{cases}$

2. $\begin{cases} -6x + \frac{1}{2}y = 4 \\ y = 4 \end{cases}$

3. $\begin{cases} 5x - y = 17 \\ x = y + 1 \end{cases}$

Learning Goals

- Solve systems of two linear equations.
- Solve systems of equations involving one linear and one quadratic equation.
- Write and solve systems of three linear equations in three variables by using substitution.
- Write and solve systems of three linear equations in three variables by using Gaussian elimination.

Key Term

- Gaussian elimination

You know how to solve a system of linear equations in two variables graphically and algebraically. How can you use similar methods to solve systems of equations involving a linear and a quadratic equation, or three linear equations?

GETTING STARTED

Which Fare Is Fair?

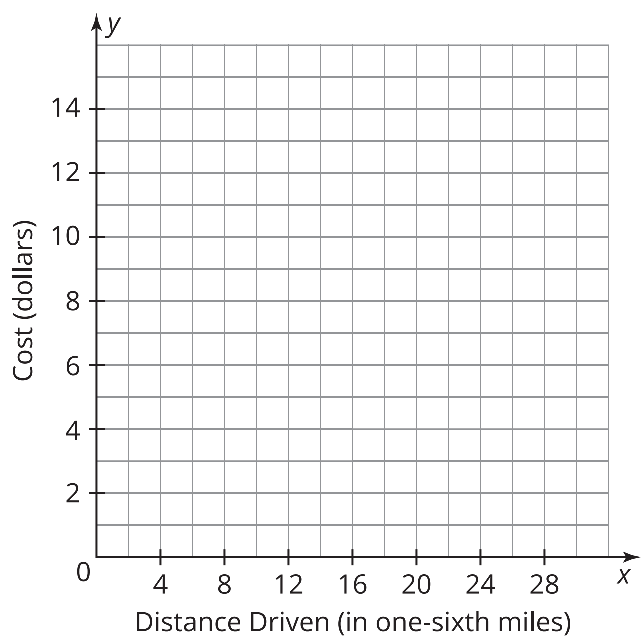
You would like to take a taxi to the airport. There are two local taxi companies. Friendly's Cab Company charges $2.60 plus $0.20 per one-sixth of a mile driven. Anderson Taxi, Inc. charges $5.00 plus $0.10 per one-sixth of a mile driven.

1. Write a system of two linear equations in two variables to represent this problem situation. Be sure to define your variables.

2. Graph the system of equations.

Remember:

The solution to a system of linear equations occurs when the values of the variables satisfy all of the linear equations.



3. Estimate the solution to the system of equations. Justify your reasoning.

4. Solve the system of linear equations algebraically.

5. What does the solution mean in terms of the problem situation?

6. Suppose that Anderson Taxi, Inc. decides to increase its fare to $0.20 per one-sixth mile driven. Write a new system of equations to reflect the increased fare. When will the cost of using the two taxi companies be equal for the same number of miles? Explain your reasoning.

7. Think about the graphs of different systems of two linear equations.

a. Describe the different ways in which the two graphs can intersect, and provide a sketch of each case.

b. How does this relate to the number of solutions to a system of two linear equations?

Think about:

What are some different methods you have learned to solve a system of linear equations in two variables algebraically?

ACTIVITY

1.1

Solving Systems with a Linear and a Quadratic Equation

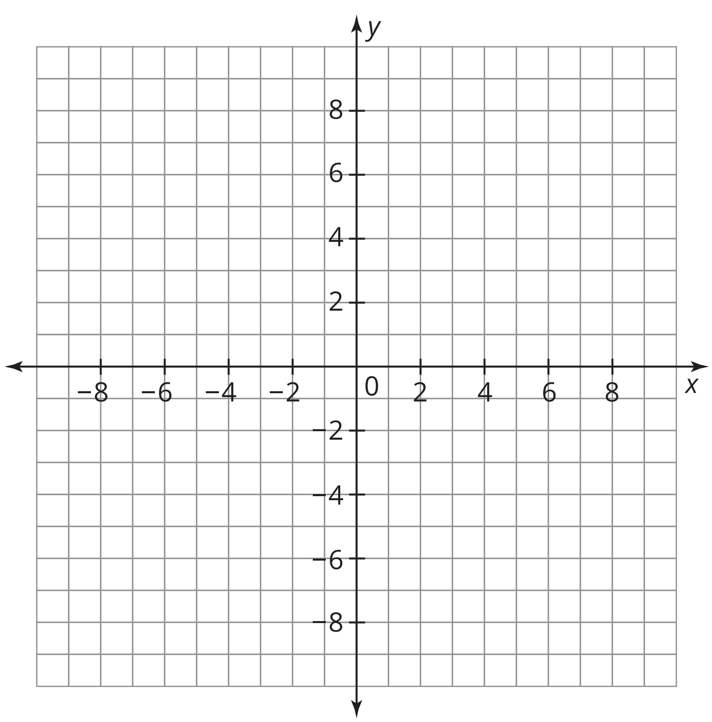
A system of equations can also involve non-linear equations, such as quadratic equations. The methods for solving a system of non-linear equations are similar to methods for solving a system of linear equations.

1. Consider the system of a linear equation and a quadratic equation: $\begin{cases} x^2 - y = 8 \\ 2x - y = 5 \end{cases}$.

a. Use substitution to write a new equation that can be used to solve this system.

b. Solve the resulting equation for *x*.

c. Calculate the corresponding value(s) for *y*. Determine the solution(s) to the system of equations.



d. Graph and label each equation of the system and identify the point(s) of intersection.

e. What do you notice about the solutions that you determined algebraically and graphically?

2. Think about the graphs of a linear equation and a quadratic equation.

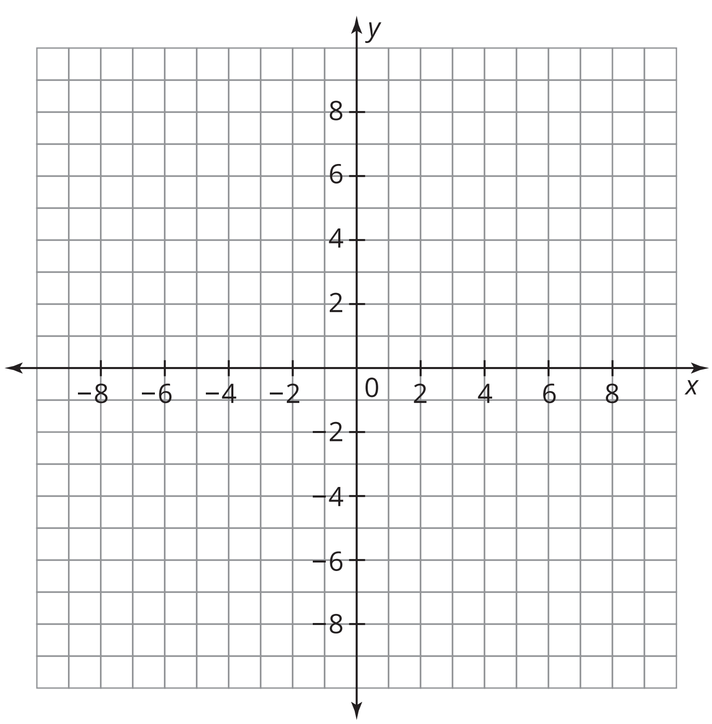
a. Describe the different ways in which the two graphs can intersect, and provide a sketch of each case.

b. How does this relate to the number of solutions to a system of one linear and one quadratic equation?

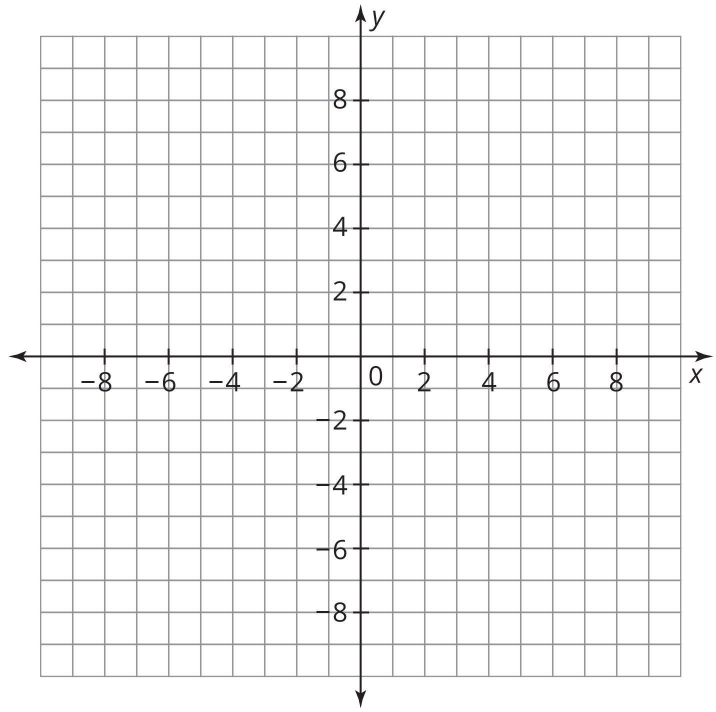
3. Can a system of a linear equation and a quadratic equation ever have infinitely many solutions? Explain your reasoning.

4. Solve each system of two equations in two variables algebraically. Then verify the solution graphically.

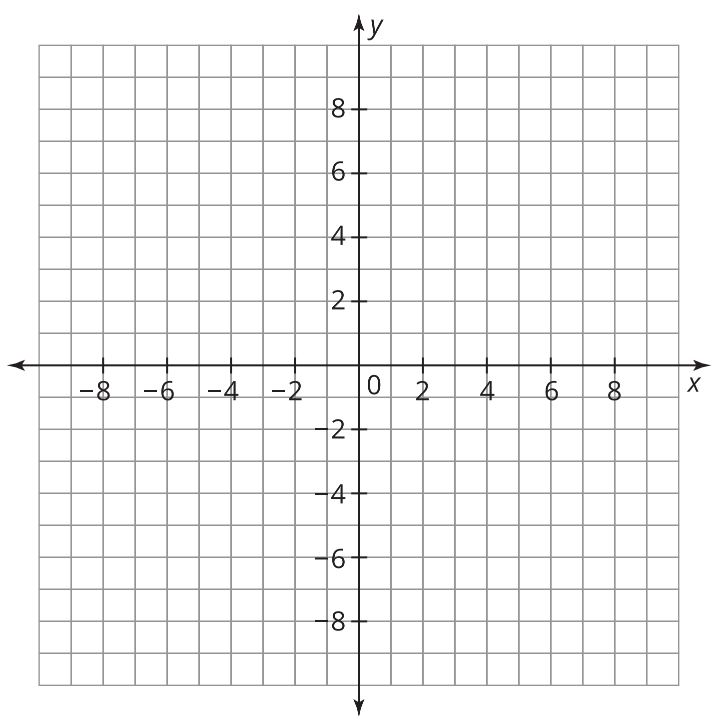
a. $\begin{cases} y = -x^2 + 2x + 8 \\ y = 3x + 2 \end{cases}$



b. $\begin{cases} y = x^2 - 3x - 4 \\ y = x - 8 \end{cases}$



c. $\begin{cases} y = 2x^2 + 4x + 3 \\ y = 4x - 1 \end{cases}$



5. Simon and his sister are playing a guessing game. Simon tells his sister that he is thinking of two positive numbers. The first number minus the second number is 15. The square of the first number minus 20 times the second number is equal to 300.

   a. Write a system of one linear and one quadratic equation to represent the two numbers. Be sure to define your variables.

   b. Solve the system of equations.

   c. What are the two numbers that Simon is thinking of? Explain your reasoning.

ACTIVITY
1.2 Modeling with Systems of Equations

A photographer specializes in taking senior portraits. She records the amount of revenue she earns for each senior portrait package that she sells.

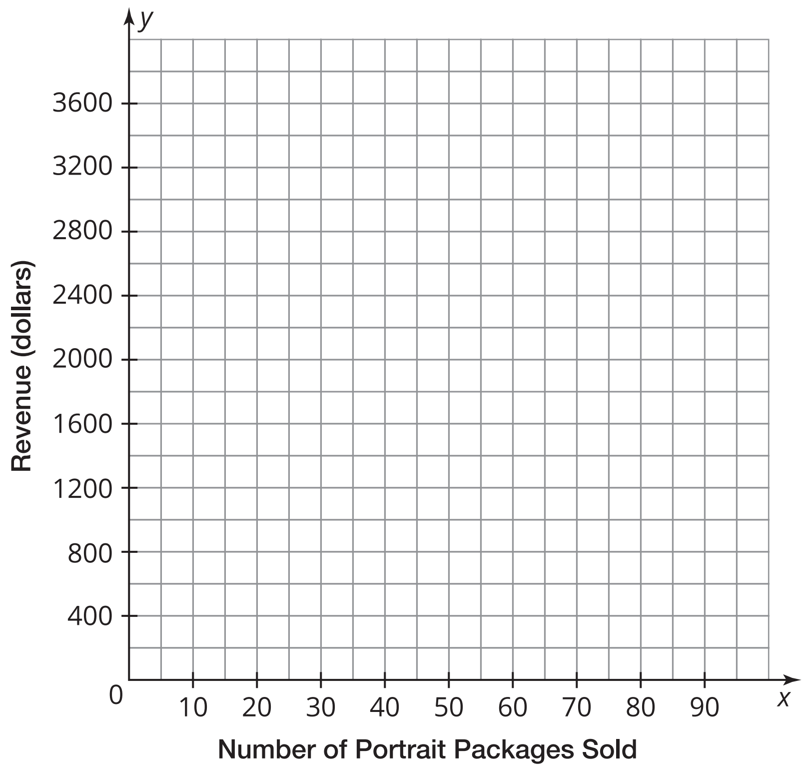
Number of Portrait Packages Sold	Revenue (dollars)
5	575
10	1100
15	1575
20	2000
25	2375
30	2700
35	2975
40	3200

1. **Use quadratic regression to write an equation to model the photographer's revenue. Be sure to define your variables.**

2. **Each month, the photographer must keep track of her costs and revenue. Her costs consist of a fixed amount of $1400, which includes rent, utilities, and workers' salaries, as well as $30 per package to print the portraits. Write a linear equation to model the photographer's costs.**

3. Write a system of one linear and one quadratic equation to represent this problem situation.

4. Graph the system of equations.



5. Solve the system of equations.

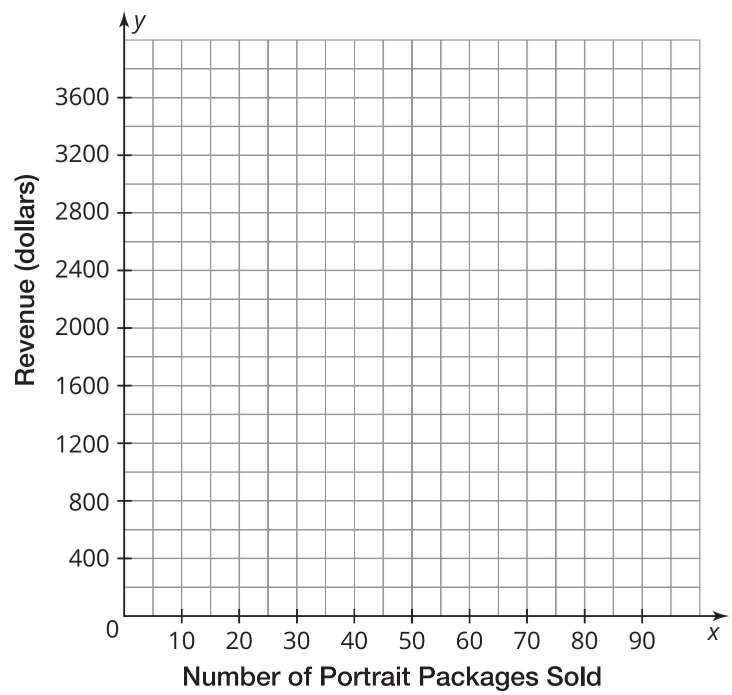
6. Explain the solution(s) in terms of this problem situation.

7. Determine the amount of portrait packages that the photographer needs to sell in order to make a profit. Is this solution reasonable in terms of the problem situation?

Due to the rising costs of running a business, the photographer anticipates fixed costs in the next year to be $2400 per month, whereas the costs for portrait packages will increase to $35 per portrait package.

8. Write a new system of equations to reflect the changes in the cost.

9. Graph the system.



10. Determine the solution(s) to the system of equations.

11. Explain the solution(s) in terms of the problem situation. Are the solutions reasonable?

ACTIVITY **1.3**

Solving Systems of Three Linear Equations

You can solve systems of three linear equations in three variables by using substitution. First, solve one equation for a variable and then substitute that expression into the other two equations. This reduces the system of three equations in three variables to a system of two equations in two variables, which you can then solve using any method.

1. As a fundraising event, a club sold tickets to a special viewing of a classic movie. The club sold all 800 seats in the school's auditorium. The tickets were three different prices: $2.50 for children under 12 years old, $3.50 for youth between 12 and 18 years old, and $5.00 for adults. The total amount of money taken in was $2937.50, and there were 4 times as many youth tickets as children's tickets sold.

a. Write a system of three linear equations in three variables to represent this situation. Be sure to define your variables.

Remember:

You can use graphing, substitution, or linear combinations to solve a system of two equations.

b. The equation $t = 4c$ is already solved for a variable. Substitute $4c$ for t in the other two equations to create a system of two equations in two variables.

c. Solve the resulting system. Explain what your solution represents in terms of the problem situation.

2. Allison and Emily were asked to solve this system of three linear equations using substitution. Analyze their methods.

$$\begin{cases} x + y + z = -2 \\ 2x - 3y + 2z = -14 \\ 4x + 3y - z = 5 \end{cases}$$

Allison

$x + y + z = -2$

$x = -2 - y - z$

$$\begin{aligned} 2x - 3y + 2z &= -14 \\ 2(-2 - y - z) - 3y + 2z &= -14 \\ -4 - 2y - 2z - 3y + 2z &= -14 \\ -4 - 5y &= -14 \\ -5y &= -10 \end{aligned}$$

$$\begin{aligned} 4x + 3y - z &= 5 \\ 4(-2 - y - z) + 3y - z &= 5 \\ -8 - 4y - 4z + 3y - z &= 5 \\ -8 - y - 5z &= 5 \\ y + 5z &= -13 \end{aligned}$$

Then I solved the system $\begin{cases} -5y = -10 \\ y + 5z = -13 \end{cases}$.

Emily

$4x + 3y - z = 5$

$4x + 3y - 5 = z$

$$\begin{aligned} 2x - 3y + 2z &= -14 \\ 2x - 3y + 2(4x + 3y - 5) &= -14 \\ 2x - 3y + 8x + 6y - 10 &= -14 \\ 10x + 3y - 10 &= -14 \\ 10x + 3y &= -4 \end{aligned}$$

$$\begin{aligned} x + y + z &= -2 \\ x + y + (4x + 3y - 5) &= -2 \\ 5x + 4y - 5 &= -2 \\ 5x + 4y &= 3 \end{aligned}$$

Then I solved the system $\begin{cases} 10x + 3y = -4 \\ 5x + 4y = 3 \end{cases}$.

a. **Describe the similarities and differences in Allison's and Emily's methods.**

b. **Demonstrate that Allison's method and Emily's method will both yield the same solution. Explain your reasoning.**

c. **Could this system have been solved using a different substitution? How do you select which variable to solve for? Explain your reasoning.**

3. **During the movie fundraising event, the concession stand at the auditorium sells popcorn, fruit, and drinks. The price of a box of popcorn is $0.50 more than the price of a piece of fruit, and the price of two drinks is $0.50 less than the price of three pieces of fruit. You order two boxes of popcorn, one piece of fruit, and three drinks, and the total comes to $7.75.**

   a. **Write a system of three linear equations in three variables to represent this situation. Be sure to define your variables.**

   b. **Calculate the price of each item. Use substitution to solve the system of three linear equations in three variables.**

ACTIVITY

1.4 Gaussian Elimination

Gaussian elimination is a method for solving linear systems of equations, named after the mathematician Carl Friedrich Gauss. It involves using linear combinations of the equations in the system to isolate one variable per equation.

The goal of Gaussian elimination is to eliminate all variables except one in each equation.

To do this, look for ways you can add or subtract two equations to eliminate one of the variables. You can multiply one or more equations by a constant before adding or subtracting.

Worked Example

Consider the system $\begin{cases} 2x + 2y + 3z = 3 \\ x + 3y + 2z = 5 \\ 3x + y + z = 5 \end{cases}$.

You can use the second equation to eliminate the x in the other two equations.

1. Multiply the second equation by −2 and add the result to the first equation.

$$-2(x + 3y + 2z = 5) \rightarrow \begin{array}{r} -2x - 6y - 4z = -10 \\ +2x + 2y + 3z = 3 \\ \hline -4y - z = -7 \end{array}$$

2. Replace the first equation in the system with the new equation.

$$\begin{cases} -4y - z = -7 \\ x + 3y + 2z = 5 \\ 3x + y + z = 4 \end{cases}$$

3. Multiply the second equation by −3 and add the result to the third equation.

$$-3(x + 3y + 2z = 5) \rightarrow \begin{array}{r} -3x - 9y - 6z = -15 \\ +3x + y + z = 4 \\ \hline -8y - 5z = -11 \end{array}$$

4. Replace the third equation in the system with the new equation.

$$\begin{cases} -4y - z = -7 \\ x + 3y + 2z = 5 \\ -8y - 5z = -11 \end{cases}$$

You have now eliminated the x-variable in all equations except the second equation. Continue in this same manner to isolate y and z.

1. **Continue to solve the system of three linear equations in three variables from the worked example using Gaussian elimination.**

   a. **Swap the first and second equations so that the equation with the *x* variable is on top.**

   b. **Multiply the second equation by −2 and add the result to the third equation.**

   c. **Multiply the resulting equation by $-\frac{1}{3}$. Replace the third equation.**

   d. **Add the third equation to the second equation. Replace the second equation.**

   e. **Multiply the second equation by $-\frac{1}{4}$.**

   f. **Multiply the second equation by −3 and add the result to the first equation. Replace the first equation.**

   g. **Multiply the third equation by −2 and add the result to the first equation. Replace the first equation.**

2. **Jamie claims that the system in Question 1 could have been solved more efficiently by starting with a different approach.**

Jamie

I multiplied the second equation by -5 and added it to the third equation.

$$\begin{cases} x + 3y + 2z = 5 \\ -4y - z = -7 \\ -8y - 5z = -11 \end{cases}$$

$$-5(-4y - z = -7) \rightarrow \begin{array}{r} 20y + 5z = 35 \\ -8y - 5z = -11 \\ \hline 12y \quad\quad = 24 \end{array}$$

a. **How is Jamie's method different from the method used in Question 1?**

b. **Complete Jamie's method. Describe your steps.**

c. **How does this solution compare to the solution you got in Question 1?**

d. **Compare the two methods. What can you determine about the order in which equations are combined when using Gaussian elimination?**

There is no one correct order in which to perform the steps to Gaussian elimination. The idea is to keep performing linear combinations until each equation contains a different isolated variable.

3. Ethan, Jackson, and Olivia are each asked to use Gaussian elimination to solve a system of three linear equations in three variables, but they each have a different idea of how to start the problem.

$$\begin{cases} -2x + y + 2z = -3 \\ -3x - y + 2z = -11 \\ 2x + y - z = 8 \end{cases}$$

Ethan

I would add equation 1 to equation 3, and replace equation 3. This would eliminate the x variable.

$$\begin{array}{r} -2x + y + 2z = -3 \\ 2x + y - z = 8 \\ \hline 2y + z = 5 \end{array}$$

$$\begin{cases} -2x + y + 2z = -3 \\ -3x - y + 2z = -11 \\ 2y + z = 5 \end{cases}$$

Jackson

I would add equation 1 to equation 2 and replace equation 2. This would eliminate the y variable.

$$\begin{array}{r} -2x + y + 2z = -3 \\ -3x - y + 2z = -11 \\ \hline -5x + 4z = -14 \end{array}$$

$$\begin{cases} -2x + y + 2z = -3 \\ -5x + 4z = -14 \\ 2x + y - z = 8 \end{cases}$$

Olivia

I would subtract equation 2 from equation 1 and replace equation 2. This would eliminate the z variable.

$$\begin{array}{r} -2x + y + 2z = -3 \\ 3x + y - 2z = 11 \\ \hline x + 2y = 8 \end{array}$$

$$\begin{cases} -2x + y + 2z = -3 \\ x + 2y = 8 \\ 2x + y - z = 8 \end{cases}$$

a. Describe the similarities and differences among all three methods.

b. Whose method is correct? Explain your reasoning.

c. Is there a different way to begin the problem than the ones that were mentioned? Justify your answer.

4. Solve the system of three linear equations in three variables from Question 3 using Gaussian elimination.

5. Colleen has 12 coins in her pocket. The mix of quarters, nickels, and dimes add up to two dollars, and she has three times as many quarters as nickels.

a. Write a system of three linear equations in three variables to represent this problem situation. Be sure to define your variables.

b. How many of each coin does Colleen have in her pocket? Solve this system using Gaussian elimination.

NOTES

TALK the TALK

You Can Have It Your Way

You can use either substitution or Gaussian elimination to solve systems of three linear equations in three variables.

1. List an advantage and a disadvantage to the substitution method.

2. List an advantage and a disadvantage to the Gaussian elimination method.

3. For each system, determine whether you would prefer to use substitution or Gaussian elimination. Justify your reasoning. Perform the first step.

a. $\begin{cases} 5x + 3y + 2z = -3 \\ 4x - 3y = 4 \\ 2x - 3y + 4z = -14 \end{cases}$

b. $\begin{cases} x + y + 2z = -7 \\ 3x + 4y - 2z = -10 \\ x - y = 7 \end{cases}$

4. Solve the system using whichever method you prefer.

$$\begin{cases} 2x + 3y - 4z = -16 \\ 2y + z = -1 \\ 3x - 4z = -9 \end{cases}$$