# 1

# Gauss in Das Haus

## Solving Systems of Equations

**MATERIALS**

Graphing technology

### Lesson Overview

Students solve systems comprised of linear and quadratic equations. They begin by solving a system of two linear equations graphically and algebraically. Students then use substitution to solve a system that is comprised of a quadratic equation and a linear equation. In each case, they use graphs to determine the number of possible solutions to that type of system. Students practice solving systems of two equations in real-world and mathematical problems. Students then solve systems of three linear equations in three variables using substitution and Gaussian elimination, both in and out of context.

### Algebra 2
### Systems of Equations and Inequalities

**(3) The student applies mathematical processes to formulate systems of equations and inequalities, uses a variety of methods to solve, and analyzes reasonableness of solutions. The student is expected to:**

(A) formulate systems of equations, including systems consisting of three linear equations in three variables and systems consisting of two equations, the first linear and the second quadratic.
(B) solve systems of three linear equations in three variables by using Gaussian elimination, technology with matrices, and substitution.
(C) solve, algebraically, systems of two equations in two variables consisting of a linear equation and a quadratic equation.
(D) determine the reasonableness of solutions to systems of a linear equation and a quadratic equation in two variables.

### ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I 3.D, 3.E, 4.B, 4.C, 5.B, 5.F, 5.G

### Essential Ideas

- A system of equations composed of two linear equations can have zero, one, or an infinite number of solutions.

- A system of equations composed of one linear equation and one quadratic equation can have zero, one, or two solutions.
- To solve a system of three linear equations in three variables, you can solve one equation for a variable, substitute that expression into the other two equations, and then solve the resulting system of two equations in two variables using any method.
- Gaussian elimination is an algorithm that can be used to solve systems of three linear equations in three variables. It involves using linear combinations of the equations in the system to isolate one variable per equation.

# Lesson Structure and Pacing: 2 Days

## Day 1

### Engage

**Getting Started: Which Fare Is Fair?**

Students use a system of two linear equations to represent a context. They solve the system graphically and algebraically. The context is then modified, which results in a system with no solutions. Students sketch the three different ways the graphs of two lines may intersect and relate them to the number of possible solutions for a system of equations.

### Develop

**Activity 1.1: Solving Systems with a Linear and a Quadratic Equation**

Students solve a system composed of a linear equation and quadratic equation both algebraically and graphically. They sketch the three different ways the graphs of a line and parabola may intersect to determine the number of solutions for the system of equations. Students then solve systems algebraically and verify their results graphically. They then write and solve a system to model a scenario.

**Activity 1.2: Modeling with Systems of Equations**

Students model a scenario with a system of equations composed of a quadratic equation and a linear equation. The quadratic equation is determined by completing a regression for a set of data.

## Day 2

**Activity 1.3: Solving Systems of Three Linear Equations**

Students write a system of three linear equations in three variables to model a scenario. They use substitution to eliminate one variable and then solve the resulting system of two linear equations. Students then analyze methods to solve another system of equations before solving another problem in context that can be modeled by a system of three linear equations.

**Activity 1.4: Gaussian Elimination**

Students are introduced to Gaussian elimination as a strategy to solve systems of equations. They analyze a worked example and follow guided questions to solve a system of three equations in three variables using this method. Through student work, students discover that there are multiple possible solution paths. They solve several systems, including one in context, using Gaussian elimination.

### Demonstrate

**Talk the Talk: You Can Have It Your Way**

Students compare the substitution method and Gaussian elimination to solve systems of three linear equations in three variables. They then are given two different systems of three linear equations in three variables and decide which method they would prefer to use to solve the system. They explain their reasoning and perform the first step. Students then use the method of their choice to solve a system of three linear equations.

ENGAGE

# Getting Started: Which Fare Is Fair?

## Facilitation Notes

In this activity, students use a system of two linear equations to represent a context. They solve the system graphically and algebraically. The context is then modified, which results in a system with no solutions. Students sketch the three different ways the graphs of two lines may intersect and relate the sketches to the number of possible solutions for a system of equations.

Have students work with a partner or in a group to complete Questions 1 through 5. Share responses as a class.

**Differentiation strategy**

To extend the activity, after students have read the introduction, have them close their books and answer the question, "For what number of miles will the charge for both companies be the same amount." This provides students the opportunity to choose their solution method (guess and check, algebra, graphing technology) and indicates which method(s) they are comfortable with.

**Misconception**

- A common response is that the point of intersection represents when both plans cost the same amount. Have student restate their response taking both variables into account, so that they acknowledge that both plans cost the same amount for the same number of miles.
- While students often realize the importance of the point of intersection, they may overlook the importance of the intervals to the left and right of this point. Ask them to interpret the meaning of both of these intervals in regard to the context. This will help them in future problems when they must interpret intervals created by more than one point of intersection.

**Questions to ask**

- What do the variables in your equations represent?
- Why does it make sense to write your equations in general form?
- Which graph has a steeper slope? Why is that the case?
- What is the meaning of each $y$-intercept?
- What method did you use to solve the system of equations algebraically?
- Did anyone use a different method? If so, explain your process.
- Why does it make sense that the point of intersection is the solution to the system of equations?

Have students work with a partner or in a group to complete Questions 6 and 7. Share responses as a class.

**As students work, look for**

Immediate recognition that the Anderson Taxi, Inc. always costs more than Friendly's Cab Company because both both companies have the same rate, but the one-time fee for Anderson Taxi, Inc. is higher.

**Differentiation strategies**

- To extend the activity, have students graph the line representing their new equation on the coordinate plane in Question 2.
- To scaffold support for all students, suggest they add a second set of arrows to their sketch showing both lines are the same to signify one line is lying on top of the other

**Questions to ask**

- How can you determine algebraically that there is no solution to the system?
- Describe the graph of this system of equations.
- How can you tell from the equations that the lines are parallel?
- How can you tell from the context that there is no solution?

## Summary

A system of two linear equations can be solved algebraically and approximated from a graph. It may have one solution, infinite solutions, or no solution.

# Activity 1.1

### Solving Systems with a Linear and a Quadratic Equation

DEVELOP

## Facilitation Notes

In this activity, students solve a system composed of a linear equation and quadratic equation both algebraically and graphically. They sketch the three different ways the graphs of a line and parabola may intersect to determine the number of solutions for the system of equations. Students then solve systems algebraically and verify their results graphically. They then write and solve a system to model a scenario.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

**Questions to ask**

- What algebraic steps did you use to write your equation to solve the system?
- Why does the solution process always result in a quadratic equation?
- What method did you use to solve the resulting quadratic equation? Why did you choose that method?
- Which equations did you use to determine the value of $y$? Does it make a difference? Explain.
- How many solutions are possible when solving a quadratic equation?
- How many solutions are possible for a system involving a linear equation and a quadratic equation?
- How can you be sure that a parabola and line in your sketch will never intersect?

**Differentiation strategy**

To extend the activity, have students relate the different possible results from using the Quadratic Formula to their responses to Question 2.

Have students work with a partner or in a group to complete Questions 4 and 5. Share responses as a class.

**As students work, look for**

Whether or not students remember to solve for the $y$-coordinate of their answers.

**Questions to ask**

- Were you able to determine the number of solutions to the system from the equations? Why or why not?
- What steps did you take to solve the system algebraically?
- Did anyone use a different method? If so, explain your process.
- What issues might arise from using the graphing method?
- For Question 5, did you set the equations equal to one another to solve this system? Why or why not?
- Why does Question 5 only have one solution when you determined two solutions when solving it algebraically?

## Summary

The methods for solving a system of non-linear equations are similar to methods for solving a system of linear equations. A system of one linear and one quadratic equation always results in a quadratic equation that may have two solutions, one solution, or no solutions.

# Activity 1.2
## Modeling with Systems of Equations

## Facilitation Notes

In this activity, students model a scenario with a system of equations composed of a quadratic equation and a linear equation. The quadratic equation is determined by completing a regression for a set of data.

Ask a student to read the introduction aloud. Discuss the context as a class.

**Differentiation strategy**

To extend the activity, have students calculate second differences to demonstrate that the data is quadratic.

**Questions to ask**

- Why do you think a quadratic equation is used to model this context?
- What are the independent and dependent variables?
- What does the solution to the system of equations represent?

Have students work with a partner or in a group to complete Questions 1 through 7. Share responses as a class.

**Misconception**

Students may think that the photographer will make a profit only if she sells exactly 20 or 70 photo packages. Discuss how the points of intersection are the solution to the system of equations; however, further interpretation is required to answer Questions 6 and 7.

**Questions to ask**

- Identify the intersection points using units.
- Explain what is happening in each of the three intervals of the graph separated by the intersection points.
- How do the intersection points relate to the context?
- The intersection points could be described as *break- even points*. Explain why that description makes sense in terms of the context.
- Explain your process to solve the system of equations algebraically.
- Why is the graph helpful even though the points of intersection can be determined algebraically?
- How is interpreting the solution to a system involving a quadratic equation and linear equation different than interpreting a solution for a system involving a pair of linear equations.
- What is the number of portrait packages for which the photographer would earn the largest profit?

**Differentiation strategies**

To extend the activity,

- Have students write the solution to Question 7 using inequality notation.

- Ask students how they could use the equations and/or graph to determine the largest profit that the photographer can earn.

Have students work with a partner or in a group to complete Questions 8 through 11. Share responses as a class.

**Questions to ask**

- Explain how the graph of the new linear equation relates to the parabola.
- What would happen if you tried to solve this system of equations algebraically?
- When solving a system of equations, would you prefer to solve it algebraically or graphically first? Explain your reasoning.

**Differentiation strategy**

To extend the activity, have students solve the system of equations algebraically using the Quadratic Formula.

## Summary

A real-world situation involving costs and revenue may be modeled by a system with a linear and a quadratic equation. The solution(s) to the system must be interpreted in terms of the problem situation.

# Activity 1.3

### Solving Systems of Three Linear Equations

## Facilitation Notes

In this activity, students write a system of three linear equations in three variables to model a scenario. They use substitution to eliminate one variable and then solve the resulting system of two linear equations. Students then analyze different methods to solve another system of equations before solving another problem in context that can be modeled by a system of three linear equations.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

**Differentiation strategy**

To scaffold support, work through Question 1 together. Have them take general notes alongside their work for future reference.

**Questions to ask**

- How do you know what variable to replace by substitution?
- What is the purpose of this substitution step?

- How can you solve the remaining system of two linear equations?
- Do you have to use substitution to solve the remaining system of two linear equations?
- Once you solve the system of two linear equations, how do you solve for the third variable?
- How can you check that your answer is correct?

Have students work with a partner or in a group to complete Question 2. Share responses as a class.

**Differentiation strategies**

- As an alternative grouping strategy, use the jigsaw strategy for Question 2. Assign students into groups of four. Have half the groups analyze Allison's work and the other half analyze Emily's work. Ask students to completely solve the systems for all three variables. Then re-assign students to groups of four, with two members who analyzed each girl's work. Have each pair of group members teach their strategy to the other group members. Then, have group members respond to the following questions.
- To scaffold support for all students, suggest they use colored pencils and boxes around substituted expressions to make the substitution step more explicit.

**Questions to ask**

- Why do you think neither girl chose the second equation?
- Does it matter what equation and what variable you choose? Are some choices easier than others? Explain.
- Why does one of Allison's equations only have the variable $y$? What happened to $x$?
- What method did you use to solve the resulting system of two equations?
- How do you think you could solve a system with four linear equations with four variables?

Have students work with a partner or in a group to complete Question 3. Share responses as a class.

**Questions to ask**

- What is different about this system of equations?
- What variable did you eliminate through substitution?
- What do you do if the variable you are substituting for is not present in the equation?
- Did anyone use a different variable? If so, why?
- Explain the steps you used to solve the system.
- Explain how your answers make sense for the given context.

## Summary

A system of three linear equations in three variables can be solved by using substitution to eliminate one variable and then solving the resulting system of two linear equations.

# Activity 1.4
## Gaussian Elimination

## Facilitation Notes

In this activity, students are introduced to Gaussian elimination as a strategy to solve systems of equations. They analyze a worked example and follow guided questions to solve a system of three equations in three variables using this method. Through student work, students discover that there are multiple possible solution paths. They solve several systems, including one in context, using Gaussian elimination.

Ask a student to read the introduction aloud. Discuss as a class.

Have students work with a partner or in a group to analyze the worked example and complete Question 1. Share responses as a class.

**Differentiation strategy**

To scaffold support, analyze the worked example and work through Question 1 together.

**Questions to ask**

- How do you know what value to multiply the equation by?
- Which variable will be eliminated by this step?
- Could a different multiplication step be used instead?
- Why might you want to look for a variable that has a coefficient of 1?

Have students work with a partner or in a group to complete Questions 2 through 4. Share responses as a class.

**Questions to ask**

- Why did Jamie decide to multiply by $-5$?
- Which variable will be eliminated by this step?
- Explain why neither Ethan nor Jackson needed to multiply first to eliminate a variable.
- Which variable would you have eliminated first? Why?
- How can you check that your solution is correct?

Have students work with a partner or in a group to complete Question 5. Share responses as a class.

**As students work, look for**
Whether students expressed the equation in terms of dollars, $0.25q + 0.10d + 0.05n = 2$, or in terms of cents, $25q + 10d + 5n = 200$.

**Misconception**
Students may misinterpret *three times as many quarters as nickels* as $3q = n$. Suggest students substitute values in the equation to demonstrate the error in their thinking. For example, if Colleen has 3 quarters, how many nickels should she have? Is this what your equation demonstrates? If not, how can you modify it? Explain that since Colleen has more quarters than nickels, the number of nickels must be multiplied by 3 to equal the number of quarters.

**Questions to ask**

- How did you know which equation involving all three variables should have the value of each coin as the coefficient for each variable?
- Which equation models the fact that Colleen has 12 coins in her pocket?
- How did you represent the fact that Colleen has three times as many quarters as nickels?
- Why did you rewrite the equation $q = 3n$ as $q - 3n = 0$?

## Summary

To use Gaussian elimination to solve a system of equations, look for ways to add or subtract two equations to eliminate one of the variables. Then multiply one or more equations by a constant before adding or subtracting.

# Talk the Talk: You Can Have It Your Way

DEMONSTRATE

## Facilitation Notes

In this activity, students compare the substitution method and Gaussian elimination to solve systems of three linear equations in three variables, and list an advantage and disadvantage of each method. They then are given two different systems of three linear equations in three variables. They explain their reasoning and perform the first step. Students then use the method of their choice to solve a system of three linear equations.

Have students work with a partner or in a group to complete Questions 1 through 4. Share responses as a class.

**Differentiation strategy**

To scaffold support, list advantages and disadvantages of each method, suggest they answer Question 3 first to generate ideas.

**Questions to ask**

- How can you tell by looking at a system of equations which method would be most efficient?
- When looking at a system of equations, does whether or not there are any coefficients that are 1 influence your decision? If so, explain why.
- Why did you decide to use Gaussian elimination to solve this system?
- Which variable did you decide to eliminate first?
- What steps did you use to eliminate that variable?
- Why did you decide to use substitution to solve this system?
- Which variable did you isolate? Why did you select that variable?
- Explain your substitution step.
- What method did you use to solve this system? Explain why you made that choice.
- Did anyone choose a different method? If so, explain why you made that choice.
- How do you know that your answer is correct?

## Summary

You can use either substitution or Gaussian elimination to solve systems of three linear equations in three variables.

# 1

# Gauss in Das Haus

## Solving Systems of Equations

### Warm Up

Use substitution to solve each system of equations.

1. $\begin{cases} 2x + 3y = 8 \\ x = -2 \end{cases}$

2. $\begin{cases} -6x + \frac{1}{2}y = 4 \\ y = 4 \end{cases}$

3. $\begin{cases} 5x - y = 17 \\ x = y + 1 \end{cases}$

### Learning Goals

- Solve systems of two linear equations.
- Solve systems of equations involving one linear and one quadratic equation.
- Write and solve systems of three linear equations in three variables by using substitution.
- Write and solve systems of three linear equations in three variables by using Gaussian elimination.

### Key Term

- Gaussian elimination

You know how to solve a system of linear equations in two variables graphically and algebraically. How can you use similar methods to solve systems of equations involving a linear and a quadratic equation, or three linear equations?



### Warm Up Answers

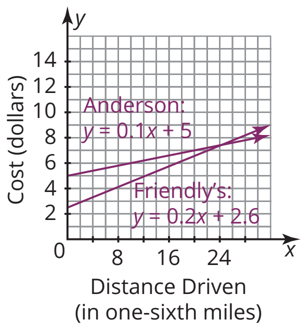
1. $(-2, 4)$
2. $\left(-\frac{1}{3}, 4\right)$
3. $(4, 3)$

## Answers

1. Let $x$ = the number of one-sixth miles driven.

   Let $y$ = the total cost of the taxi ride.

   $$\begin{cases} y = 0.20x + 2.60 \\ y = 0.10x + 5.00 \end{cases}$$

2. 


3. The solution to the system of equations is $x = 24$. This is the $x$-coordinate of the point where the two graphs intersect. The $y$-coordinate of this point is between 7 and 8.

GETTING STARTED

## Which Fare Is Fair?

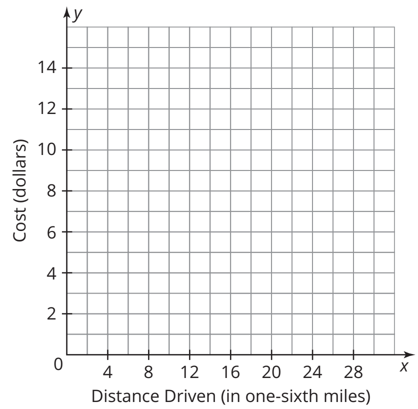
You would like to take a taxi to the airport. There are two local taxi companies. Friendly's Cab Company charges $2.60 plus $0.20 per one-sixth of a mile driven. Anderson Taxi, Inc. charges $5.00 plus $0.10 per one-sixth of a mile driven.

**1. Write a system of two linear equations in two variables to represent this problem situation. Be sure to define your variables.**

**2. Graph the system of equations.**

Remember:

The solution to a system of linear equations occurs when the values of the variables satisfy all of the linear equations.




**3. Estimate the solution to the system of equations. Justify your reasoning.**



### ELL Tip

Make sure students are familiar with what a taxi or taxicab is by comparing it to other similar transportation services.

4. **Solve the system of linear equations algebraically.**

5. **What does the solution mean in terms of the problem situation?**

**Think about:**

What are some different methods you have learned to solve a system of linear equations in two variables algebraically?

6. **Suppose that Anderson Taxi, Inc. decides to increase its fare to \$0.20 per one-sixth mile driven. Write a new system of equations to reflect the increased fare. When will the cost of using the two taxi companies be equal for the same number of miles? Explain your reasoning.**

7. **Think about the graphs of different systems of two linear equations.**

   a. **Describe the different ways in which the two graphs can intersect, and provide a sketch of each case.**

   b. **How does this relate to the number of solutions to a system of two linear equations?**



## Answers

4. The solution to the system of equations is (24, 7.4).

   Using substitution:

   $0.20x + 2.60 = 0.10x + 5.00$

   $0.10x = 2.40$

   $x = 24$

Substitute $x = 24$ into one of the linear equations:

$y = 0.20(24) + 2.60$

$y = 4.80 + 2.60$

$y = 7.40$

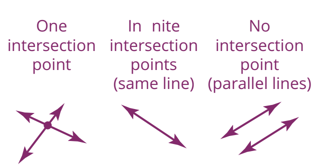
5. The point of intersection is (24, 7.4). This means that if the distance to the airport is 24 one-sixth miles, or 4 miles, then the cost will be the same using either taxi company.
   If the airport is less than 4 miles away, you should use Friendly's Cab Company. If the airport is more than 4 miles away, you should use Anderson Taxi, Inc. If the airport is exactly 4 miles away, it does not matter which taxi company you use.

6. If Anderson Taxi, Inc. increases its fare to \$0.20 per one-sixth mile driven, the two taxi companies will never be equal for the same number of miles. The new system of equations is

   $$\begin{cases} y = 0.20x + 2.60 \\ y = 0.20x + 5.00 \end{cases}$$

   The slopes of the linear equations are the same, but the $y$-intercepts are not. The graphs will never intersect. Therefore, there is no solution to this system.

7a. The graph of two linear equations can intersect each other in three ways.
The graphs can intersect at one point, at an infinite number of points, or not at all.




7b. The solution to a system of equations occurs when the values of the variables satisfy all of the linear equations. Solutions to a system of two linear equations are represented as intersection points of the graphs of the equations. Therefore, a system of two linear equations can have 1 solution, infinite solutions, or no solution.

## Answers

1a. $2x - 5 = x^2 - 8$

1b. $2x - 5 = x^2 - 8$

$0 = x^2 - 8 - 2x + 5$

$0 = x^2 - 2x - 3$

$0 = (x - 3)(x + 1)$

$x = 3$ or $x = -1$

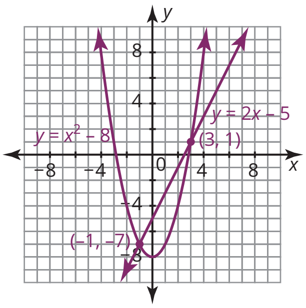
1c. The solutions to the system of equations are (3, 1) and (−1, −7).
For $x = 3$:

$y = 2(3) - 5 = 1$

For $x = -1$:

$y = 2(-1) - 5 = -7$

1d.




1e. The solutions are the same when determined algebraically and graphically.

ACTIVITY **1.1**

# Solving Systems with a Linear and a Quadratic Equation

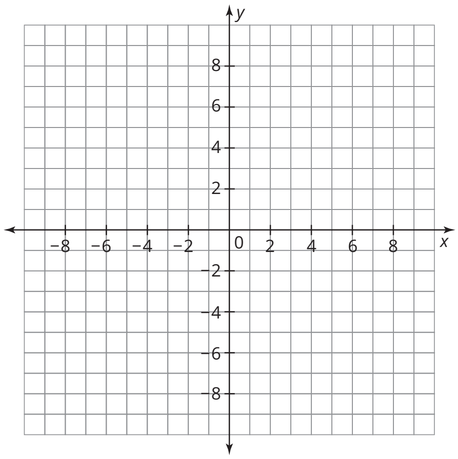
A system of equations can also involve non-linear equations, such as quadratic equations. The methods for solving a system of non-linear equations are similar to methods for solving a system of linear equations.

**1. Consider the system of a linear equation and a quadratic equation:** $\begin{cases} x^2 - y = 8 \\ 2x - y = 5 \end{cases}$.

**a. Use substitution to write a new equation that can be used to solve this system.**

**b. Solve the resulting equation for *x*.**

**c. Calculate the corresponding value(s) for *y*. Determine the solution(s) to the system of equations.**




**d. Graph and label each equation of the system and identify the point(s) of intersection.**

**e. What do you notice about the solutions that you determined algebraically and graphically?**

4 • TOPIC 1: Extending Linear Relationships

2. **Think about the graphs of a linear equation and a quadratic equation.**

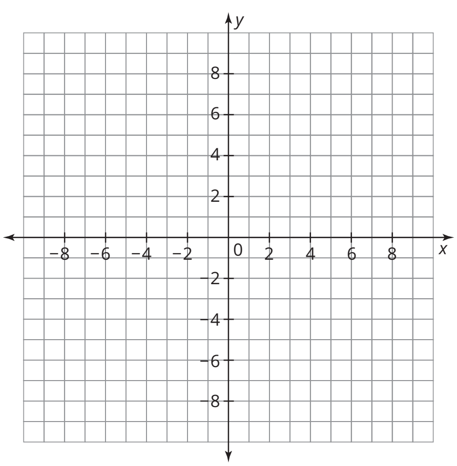
   a. **Describe the different ways in which the two graphs can intersect, and provide a sketch of each case.**

   b. **How does this relate to the number of solutions to a system of one linear and one quadratic equation?**

3. **Can a system of a linear equation and a quadratic equation ever have infinitely many solutions? Explain your reasoning.**

4. **Solve each system of two equations in two variables algebraically. Then verify the solution graphically.**

   a. $\begin{cases} y = -x^2 + 2x + 8 \\ y = 3x + 2 \end{cases}$






4a. The solutions to the system are (−3, −7) and (2, 8).

$$3x + 2 = -x^2 + 2x + 8$$

$$3x + 2 + x^2 - 2x - 8 = 0$$

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0$$

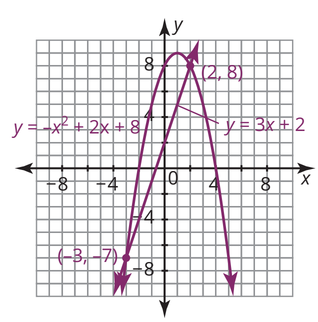
$x = -3$ or $x = 2$

For $x = -3$:

$y = 3(-3) + 2 = -7$

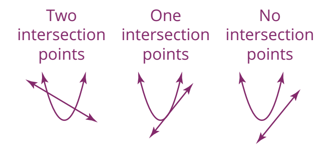
For $x = 2$:

$y = 3(2) + 2 = 8$




## Answers

2a. The graph of a linear equation can intersect the graph of a quadratic equation in three ways. The graphs can intersect at two points, at one point, or not at all.




2b. The solution to a system of equations occurs when the values of the variables satisfy all of the equations. Solutions to a system of one quadratic and one linear equation are represented as intersection points of the graphs of the equations. Therefore, a system of one linear and one quadratic equation can have 2 solutions, 1 solution, or no solutions.

3. No. A system of a linear equation and a quadratic equation can never have infinitely many solutions. For that to be true, they would have to be the same equation. Since one is linear and one is quadratic, that is impossible.

## Answers

4b. The solution to the system is (2, −6).

$$x^2 - 3x - 4 = x - 8$$

$$x^2 - 3x - 4 - x + 8 = 0$$

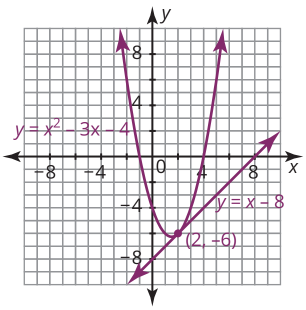
$$x^2 - 4x + 4 = 0$$

$$(x - 2)(x - 2) = 0$$

$$x = 2$$

For $x = 2$:

$y = 2 - 8 = -6$




4c. The system of equations has no solution.

$$2x^2 + 4x + 3 = 4x - 1$$

$$2x^2 + 4x + 3 - 4x + 1 = 0$$

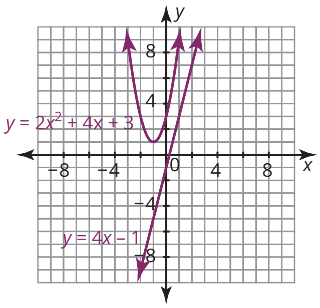
$$2x^2 + 4 = 0$$

$$2x^2 = -4$$

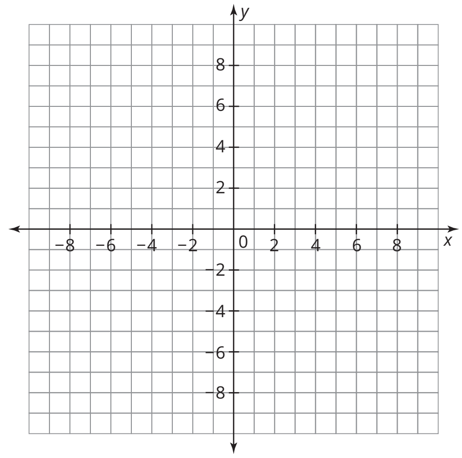
$$x^2 = -2$$

$$x = \pm\sqrt{-2}$$

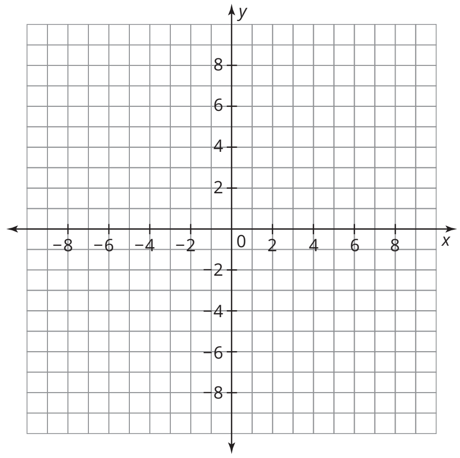
This equation has no real solutions.




b. $\begin{cases} y = x^2 - 3x - 4 \\ y = x - 8 \end{cases}$




c. $\begin{cases} y = 2x^2 + 4x + 3 \\ y = 4x - 1 \end{cases}$




6 • TOPIC 1: Extending Linear Relationships

5. **Simon and his sister are playing a guessing game. Simon tells his sister that he is thinking of two positive numbers. The first number minus the second number is 15. The square of the first number minus 20 times the second number is equal to 300.**

   a. **Write a system of one linear and one quadratic equation to represent the two numbers. Be sure to define your variables.**

   b. **Solve the system of equations.**

   c. **What are the two numbers that Simon is thinking of? Explain your reasoning.**



## Answers

5a. Let $x$ = the first number and $y$ = the second number.

$$\begin{cases} x - y = 15 \\ x^2 - 20y = 300 \end{cases}$$

5b. The solution to the system of equations are (0, −15) and (20, 5).

$x^2 - 20(x - 15) = 300$

$x^2 - 20x + 300 = 300$

$x^2 - 20x = 0$

$x(x - 20) = 0$

$x = 0$ or $x = 20$

For $x = 0$:

$y = 0 - 15 = -15$

For $x = 20$:

$y = 20 - 15 = 5$

5c. The first number is 20 and the second number is 5. The numbers 0 and −15 also satisfy the system of equations, but Simon said he was thinking of two positive numbers, and neither of these numbers are positive.

## Answers

1. Let $x$ = the number of senior portraits sold and $y$ = the revenue in dollars.

   $y = -x^2 + 120x$

2. $y = 30x + 1400$

ACTIVITY **1.2**

# Modeling with Systems of Equations

A photographer specializes in taking senior portraits. She records the amount of revenue she earns for each senior portrait package that she sells.

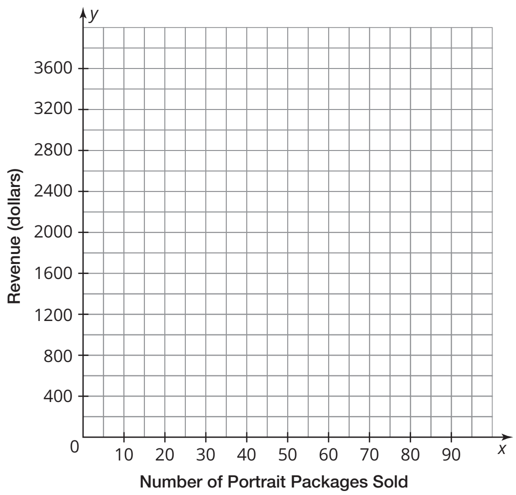
| Number of Portrait Packages Sold | Revenue (dollars) |
|---|---|
| 5 | 575 |
| 10 | 1100 |
| 15 | 1575 |
| 20 | 2000 |
| 25 | 2375 |
| 30 | 2700 |
| 35 | 2975 |
| 40 | 3200 |

**1. Use quadratic regression to write an equation to model the photographer's revenue. Be sure to define your variables.**

**2. Each month, the photographer must keep track of her costs and revenue. Her costs consist of a fixed amount of $1400, which includes rent, utilities, and workers' salaries, as well as $30 per package to print the portraits. Write a linear equation to model the photographer's costs.**

**3. Write a system of one linear and one quadratic equation to represent this problem situation.**

**4. Graph the system of equations.**




**5. Solve the system of equations.**

**6. Explain the solution(s) in terms of this problem situation.**

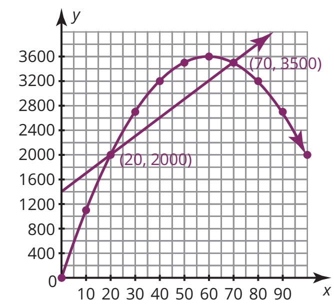
**7. Determine the amount of portrait packages that the photographer needs to sell in order to make a profit. Is this solution reasonable in terms of the problem situation?**



## Answers

3. $\begin{cases} y = -x^2 + 120x \\ y = 30x + 1400 \end{cases}$

4.




5. The system has two solutions: (70, 3500) and (20, 2000).

   $-x^2 + 120x = 30x + 1400$

   $0 = x^2 - 120x + 30x + 1400$

   $0 = x^2 - 90x + 1400$

   $0 = (x - 70)(x - 20)$

   $x = 70$ or $x = 20$

   For $x = 70$:

   $y = 30(70) + 1400 = 3500$

   For $x = 20$:

   $y = 30(20) + 1400 = 2000$

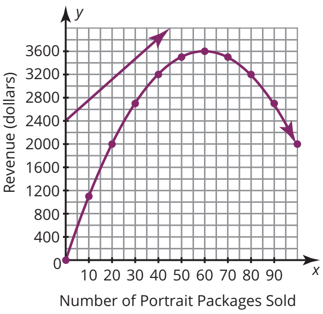
6. The photographer will break even if she sells 20 portrait packages, because the cost and revenue will both be equal to $2000. The photographer will also break even if she sells 70 portrait packages, because the cost and revenue will both be equal to $3500.

7. The photographer needs to sell between 21 and 69 portrait packages in order to make a profit. Yes, the solution is reasonable. If the photographer sells too few or too many portrait packages, the operation costs become greater than the profit that she makes from them.

## Answers

8. $\begin{cases} y = -x^2 + 120x \\ y = 35x + 2400 \end{cases}$

9.



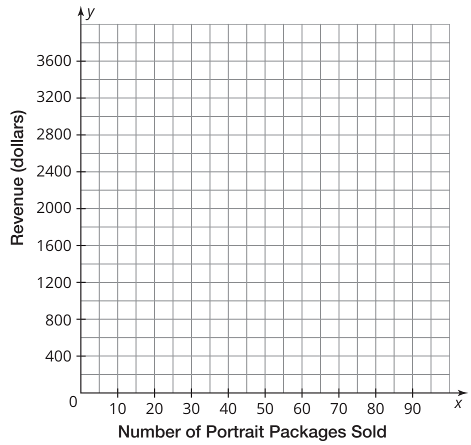

10. The graphs do not intersect.
    There is no real solution to this system of equations.

11. With the increased cost of business, the photographer will never break even. The profit is always less than the cost.
    Yes, the solution is reasonable. The cost of business increased to a point that the photographer will no longer make a profit.

Due to the rising costs of running a business, the photographer anticipates fixed costs in the next year to be $2400 per month, whereas the costs for portrait packages will increase to $35 per portrait package.

**8. Write a new system of equations to reflect the changes in the cost.**

**9. Graph the system.**




**10. Determine the solution(s) to the system of equations.**

**11. Explain the solution(s) in terms of the problem situation. Are the solutions reasonable?**

ACTIVITY 1.3

## Solving Systems of Three Linear Equations

You can solve systems of three linear equations in three variables by using substitution. First, solve one equation for a variable and then substitute that expression into the other two equations. This reduces the system of three equations in three variables to a system of two equations in two variables, which you can then solve using any method.

**1. As a fundraising event, a club sold tickets to a special viewing of a classic movie. The club sold all 800 seats in the school's auditorium. The tickets were three different prices: $2.50 for children under 12 years old, $3.50 for youth between 12 and 18 years old, and $5.00 for adults. The total amount of money taken in was $2937.50, and there were 4 times as many youth tickets as children's tickets sold.**

**a. Write a system of three linear equations in three variables to represent this situation. Be sure to define your variables.**

**b. The equation $t = 4c$ is already solved for a variable. Substitute $4c$ for $t$ in the other two equations to create a system of two equations in two variables.**

**c. Solve the resulting system. Explain what your solution represents in terms of the problem situation.**

**Remember:**

You can use graphing, substitution, or linear combinations to solve a system of two equations.



## Answers

1a. Let $c$ = the number of children's tickets, $t$ = the number of youth tickets, and $a$ = the number of adult tickets.

$$\begin{cases} c + t + a = 800 \\ 2.5c + 3.5t + 5a = 2937.5 \\ 4c = t \end{cases}$$

1b. $$\begin{cases} 5c + a = 800 \\ 16.5c + 5a = 2937.5 \end{cases}$$

1c. $-5(5c + a = 800)$

$$-25c - 5a = -4000$$

$$\begin{array}{r} -25c - 5a = -4000 \\ +\ 16.5c + 5a = 2937.5 \\ \hline -8.5c = -1062.5 \\ c = 125 \\ t = 4(125) \\ t = 500 \\ 125 + 500 + a = 800 \\ a = 175 \end{array}$$

In terms of the problem situation, there were 175 adult tickets, 500 youth tickets, and 125 children's tickets sold.

## Answers

2a. Their methods are similar because they both solved one equation for one variable, and then substituted that expression into the other two equations. Their methods are different because they solved for different variables in different equations. Allison solved for $x$ in the equation $x + y + z = -2$ and Emily solved for $z$ in the equation $4x + 3y - z = 5$.

2b. Allison and Emily both get $(-1, 2, -3)$ as their solution.

Allison's method:

$$\begin{cases} -5y = -10 \\ y + 5z = -13 \end{cases}$$

$$-5y = -10$$
$$y = 2$$
$$(2) + 5z = -13$$
$$5z = -15$$
$$z = -3$$
$$x = -2 - 2 - (-3)$$
$$= -1$$

Emily's method:

$$\begin{cases} 10x + 3y = -4 \\ 5x + 4y = 3 \end{cases}$$

$$10x + 3y = -4$$
$$\underline{-10x - 8y = -6}$$
$$-5y = -10$$
$$y = 2$$

$$5x + 4(2) = 3$$
$$5x + 8 = 3$$
$$5x = -5$$
$$x = -1$$
$$z = 4(-1) + 3(2) - 5$$
$$= -3$$

**2. Allison and Emily were asked to solve this system of three linear equations using substitution. Analyze their methods.**

$$\begin{cases} \mathbf{x + y + z = -2} \\ \mathbf{2x - 3y + 2z = -14} \\ \mathbf{4x + 3y - z = 5} \end{cases}$$

Allison

$x + y + z = -2$

$x = -2 - y - z$

$$2x - 3y + 2z = -14$$
$$2(-2 - y - z) - 3y + 2z = -14$$
$$-4 - 2y - 2z - 3y + 2z = -14$$
$$-4 - 5y = -14$$
$$-5y = -10$$

$$4x + 3y - z = 5$$
$$4(-2 - y - z) + 3y - z = 5$$
$$-8 - 4y - 4z + 3y - z = 5$$
$$-8 - y - 5z = 5$$
$$y + 5z = -13$$

Then I solved the system $\begin{cases} -5y = -10 \\ y + 5z = -13 \end{cases}$.

Emily

$4x + 3y - z = 5$

$4x + 3y - 5 = z$

$$2x - 3y + 2z = -14$$
$$2x - 3y + 2(4x + 3y - 5) = -14$$
$$2x - 3y + 8x + 6y - 10 = -14$$
$$10x + 3y - 10 = -14$$
$$10x + 3y = -4$$

$$x + y + z = -2$$
$$x + y + (4x + 3y - 5) = -2$$
$$5x + 4y - 5 = -2$$
$$5x + 4y = 3$$

Then I solved the system $\begin{cases} 10x + 3y = -4 \\ 5x + 4y = 3 \end{cases}$.

12 • TOPIC 1: Extending Linear Relationships

a. **Describe the similarities and differences in Allison's and Emily's methods.**

b. **Demonstrate that Allison's method and Emily's method will both yield the same solution. Explain your reasoning.**

c. **Could this system have been solved using a different substitution? How do you select which variable to solve for? Explain your reasoning.**

3. **During the movie fundraising event, the concession stand at the auditorium sells popcorn, fruit, and drinks. The price of a box of popcorn is $0.50 more than the price of a piece of fruit, and the price of two drinks is $0.50 less than the price of three pieces of fruit. You order two boxes of popcorn, one piece of fruit, and three drinks, and the total comes to $7.75.**

a. **Write a system of three linear equations in three variables to represent this situation. Be sure to define your variables.**

b. **Calculate the price of each item. Use substitution to solve the system of three linear equations in three variables.**



## Answers

2c. Yes. The equation $x + y + z = -2$ could have been solved first for either $y$ or $z$. It is more efficient to start by solving an equation that has a variable with a coefficient of 1.

3a. Let $p$ = the price of a box of popcorn, $f$ = the price of a piece of fruit, and $d$ = the price of a drink.

$$\begin{cases} p = f + 0.50 \\ 2d = 3f - 0.50 \\ 2p + f + 3d = 7.75 \end{cases}$$

3b. The price of a drink is $1.25, the price of a piece of fruit is $1.00, and the price of a box of popcorn is $1.50.

$$2(f + 0.50) + f + 3d = 7.75$$

$$2f + 1 + f + 3d = 7.75$$

$$3f + 3d = 6.75$$

$$\begin{cases} 3f + 3d = 6.75 \\ -3f + 2d = -0.50 \end{cases}$$

$$5d = 6.25$$

$$d = 1.25$$

$$2(1.25) = 3f - 0.50$$

$$2.50 = 3f - 0.50$$

$$3.00 = 3f$$

$$1.00 = f$$

$$2p + 1.00 + 3(1.25) = 7.75$$

$$2p + 1.00 + 3.75 = 7.75$$

$$2p + 4.75 = 7.75$$

$$2p = 3.00$$

$$p = 1.50$$

ACTIVITY
**1.4**

# Gaussian Elimination

**Gaussian elimination** is a method for solving linear systems of equations, named after the mathematician Carl Friedrich Gauss. It involves using linear combinations of the equations in the system to isolate one variable per equation.

The goal of Gaussian elimination is to eliminate all variables except one in each equation.

To do this, look for ways you can add or subtract two equations to eliminate one of the variables. You can multiply one or more equations by a constant before adding or subtracting.

## Worked Example

Consider the system $\begin{cases} 2x + 2y + 3z = 3 \\ x + 3y + 2z = 5 \\ 3x + y + z = 5 \end{cases}$.

You can use the second equation to eliminate the $x$ in the other two equations.

1. Multiply the second equation by −2 and add the result to the first equation.

$$-2(x + 3y + 2z = 5) \rightarrow \begin{array}{r} -2x - 6y - 4z = -10 \\ +2x + 2y + 3z = 3 \\ \hline -4y - z = -7 \end{array}$$

2. Replace the first equation in the system with the new equation.

$$\begin{cases} -4y - z = -7 \\ x + 3y + 2z = 5 \\ 3x + y + z = 4 \end{cases}$$

3. Multiply the second equation by −3 and add the result to the third equation.

$$-3(x + 3y + 2z = 5) \rightarrow \begin{array}{r} -3x - 9y - 6z = -15 \\ +3x + y + z = 4 \\ \hline -8y - 5z = -11 \end{array}$$

4. Replace the third equation in the system with the new equation.

$$\begin{cases} -4y - z = -7 \\ x + 3y + 2z = 5 \\ -8y - 5z = -11 \end{cases}$$

You have now eliminated the $x$-variable in all equations except the second equation. Continue in this same manner to isolate $y$ and $z$.

1. **Continue to solve the system of three linear equations in three variables from the worked example using Gaussian elimination.**

   a. **Swap the first and second equations so that the equation with the *x* variable is on top.**

   b. **Multiply the second equation by −2 and add the result to the third equation.**

   c. **Multiply the resulting equation by $-\frac{1}{3}$. Replace the third equation.**

   d. **Add the third equation to the second equation. Replace the second equation.**

   e. **Multiply the second equation by $-\frac{1}{4}$.**

   f. **Multiply the second equation by −3 and add the result to the first equation. Replace the first equation.**

   g. **Multiply the third equation by −2 and add the result to the first equation. Replace the first equation.**



## Answers

1a. $\begin{cases} x + 3y + 2z = 5 \\ -4y - z = -7 \\ -8y - 5z = -11 \end{cases}$

1b. $-2(-4y - z = -7) \rightarrow$

$$\begin{array}{r} 8y + 2z = 14 \\ -8y - 5z = -11 \\ \hline -3z = 3 \end{array}$$

$\begin{cases} x + 3y + 2z = 5 \\ -4y - z = -7 \\ -3z = 3 \end{cases}$

1c. $\begin{cases} x + 3y + 2z = 5 \\ -4y - z = -7 \\ z = -1 \end{cases}$

1d.
$$\begin{array}{r} -4y - z = -7 \\ z = -1 \\ \hline -4y = -8 \end{array}$$

$\begin{cases} x + 3y + 2z = 5 \\ -4y = -8 \\ z = -1 \end{cases}$

1e. $\begin{cases} x + 3y + 2z = 5 \\ y = 2 \\ z = -1 \end{cases}$

1f.
$$\begin{array}{r} x + 3y + 2z = 5 \\ -3y = -6 \\ \hline x + 2z = -1 \end{array}$$

$\begin{cases} x + 2z = -1 \\ y = 2 \\ z = -1 \end{cases}$

1g.
$$\begin{array}{r} x + 2z = -1 \\ -2z = 2 \\ \hline x = 1 \end{array}$$

$\begin{cases} x = 1 \\ y = 2 \\ z = -1 \end{cases}$

## Answers

2a. In Question 1, I multiplied the second equation by −2 to eliminate the $y$-term from the third equation. Jamie's method is different because he multiplied the second equation by a different number in order to eliminate the $z$-term from the third equation.

2b. Sample answer.

Multiply the third equation by $\frac{1}{12}$.

$$\begin{cases} x + 3y + 2z = 5 \\ -4y - z = -7 \\ y = 2 \end{cases}$$

Multiply the third equation by 4 and add the result to the second equation. Replace the second equation.

$$\begin{array}{r} 4y = 8 \\ -4y - z = -7 \\ \hline z = -1 \end{array}$$

$$\begin{cases} x + 3y + 2z = 5 \\ z = -1 \\ y = 2 \end{cases}$$

Multiply the third equation by −3 and add the result to the first equation. Replace the first equation.

$$\begin{array}{r} x + 3y + 2z = 5 \\ -3y = -6 \\ \hline x + 2z = -1 \end{array}$$

$$\begin{cases} x + 2z = -1 \\ z = -1 \\ y = 2 \end{cases}$$

Multiply the second equation by −2 and add the result to the first equation. Replace the first equation.

$$\begin{array}{r} x + 2z = -1 \\ -2z = 2 \\ \hline x = 1 \end{array}$$

$$\begin{cases} x = 1 \\ y = 2 \\ z = -1 \end{cases}$$

2c. I arrived at the same solution when I used Jamie's method.

2d. When performing Gaussian elimination, there is more than one correct way to approach the system.

**2. Jamie claims that the system in Question 1 could have been solved more efficiently by starting with a different approach.**

Jamie

I multiplied the second equation by −5 and added it to the third equation.

$$\begin{cases} x + 3y + 2z = 5 \\ -4y - z = -7 \\ -8y - 5z = -11 \end{cases}$$

$$-5(-4y - z = -7) \rightarrow \begin{array}{r} 20y + 5z = 35 \\ -8y - 5z = -11 \\ \hline 12y \quad = 24 \end{array}$$

**a. How is Jamie's method different from the method used in Question 1?**

**b. Complete Jamie's method. Describe your steps.**

**c. How does this solution compare to the solution you got in Question 1?**

**d. Compare the two methods. What can you determine about the order in which equations are combined when using Gaussian elimination?**

There is no one correct order in which to perform the steps to Gaussian elimination. The idea is to keep performing linear combinations until each equation contains a different isolated variable.

**3. Ethan, Jackson, and Olivia are each asked to use Gaussian elimination to solve a system of three linear equations in three variables, but they each have a different idea of how to start the problem.**

$$\begin{cases} -2x + y + 2z = -3 \\ -3x - y + 2z = -11 \\ 2x + y - z = 8 \end{cases}$$

**Ethan**

I would add equation 1 to equation 3, and replace equation 3. This would eliminate the x variable.

$$\begin{array}{r} -2x + y + 2z = -3 \\ 2x + y - z = 8 \\ \hline 2y + z = 5 \end{array}$$

$$\begin{cases} -2x + y + 2z = -3 \\ -3x - y + 2z = -11 \\ 2y + z = 5 \end{cases}$$

**Jackson**

I would add equation 1 to equation 2 and replace equation 2. This would eliminate the y variable.

$$\begin{array}{r} -2x + y + 2z = -3 \\ -3x - y + 2z = -11 \\ \hline -5x + 4z = -14 \end{array}$$

$$\begin{cases} -2x + y + 2z = -3 \\ -5x + 4z = -14 \\ 2x + y - z = 8 \end{cases}$$

**Olivia**

I would subtract equation 2 from equation 1 and replace equation 2. This would eliminate the z variable.

$$\begin{array}{r} -2x + y + 2z = -3 \\ 3x + y - 2z = 11 \\ \hline x + 2y = 8 \end{array}$$

$$\begin{cases} -2x + y + 2z = -3 \\ x + 2y = 8 \\ 2x + y - z = 8 \end{cases}$$

**a. Describe the similarities and differences among all three methods.**

**b. Whose method is correct? Explain your reasoning.**

**c. Is there a different way to begin the problem than the ones that were mentioned? Justify your answer.**



## Answers

3a. The three methods are similar because each of the students added two of the three equations together to eliminate a variable. The three methods are different because each student chose a different variable to eliminate first.

3b. All three methods are correct. Each method multiplied one equation by a number and then added the result to another equation.

3c. Yes. Sample answer. You could add equation 2 to equation 3 to eliminate the *y*-variable. You could multiply equation 3 by 2 and add it to equation 2 to eliminate the *z*-variable.

## Answers

4. The solution to the system is $x = 2, y = 3, z = -1$.

5a.

$$\begin{cases} 0.25q + 0.10d + 0.05n = 2 \\ q + d + n = 12 \\ q - 3n = 0 \end{cases}$$

Let $q$ = the number of quarters, $d$ = the number of dimes, and $n$ = the number of nickels.

5b. Colleen has six quarters, four dimes, and two nickels in her pocket.

**4. Solve the system of three linear equations in three variables from Question 3 using Gaussian elimination.**

**5. Colleen has 12 coins in her pocket. The mix of quarters, nickels, and dimes add up to two dollars, and she has three times as many quarters as nickels.**

**a. Write a system of three linear equations in three variables to represent this problem situation. Be sure to define your variables.**

**b. How many of each coin does Colleen have in her pocket? Solve this system using Gaussian elimination.**



### ELL Tip

Prior to having students solve Question 5, make sure they are familiar with the values of quarters, nickels, and dimes.

NOTES

## TALK the TALK

### You Can Have It Your Way

You can use either substitution or Gaussian elimination to solve systems of three linear equations in three variables.

1. **List an advantage and a disadvantage to the substitution method.**

2. **List an advantage and a disadvantage to the Gaussian elimination method.**

3. **For each system, determine whether you would prefer to use substitution or Gaussian elimination. Justify your reasoning. Perform the first step.**

   a. $\begin{cases} 5x + 3y + 2z = -3 \\ 4x - 3y = 4 \\ 2x - 3y + 4z = -14 \end{cases}$

   b. $\begin{cases} x + y + 2z = -7 \\ 3x + 4y - 2z = -10 \\ x - y = 7 \end{cases}$

4. **Solve the system using whichever method you prefer.**

   $\begin{cases} 2x + 3y - 4z = -16 \\ 2y + z = -1 \\ 3x - 4z = -9 \end{cases}$

LESSON 1: Gauss in Das Haus • 19

## Answers

1. Sample answer.
   An advantage to the substitution method is that it can be more efficient when one of the variables has a coefficient of 1.
   A disadvantage to the substitution method is that it can be more complicated and possibly involve fractions when there are no variables with a coefficient of 1.

2. Sample answer.
   An advantage to Gaussian elimination is that it generally involves working with integers.
   A disadvantage to Gaussian elimination is that the number of steps involved can introduce calculation error.

3a. Sample answer.
   I would use Gaussian elimination because the variables all have coefficients and I would end up with fractions in the equation if I tried to solve for a variable. Instead, the $y$-terms can be easily eliminated with Gaussian elimination. Multiply equation 2 by −1 and add to equation 3.

   Replace equation 3.

   $$\begin{array}{r} -4x + 3y \phantom{+ 4z} = -4 \\ 2x - 3y + 4z = -14 \\ \hline -2x \phantom{- 3y} + 4z = -18 \end{array}$$

   $$\begin{cases} 5x + 3y + 2z = -3 \\ 4x - 3y = 4 \\ -2x + 4z = -18 \end{cases}$$

3b. Sample answer.
   I would use substitution because there are variables with a coefficient of 1 and therefore they are easily solved for. First, I would solve equation 3 for $x$ and substitute the expression into the other two equations.

   $x = 7 + y$

   $(7 + y) + y + 2z = -7$

   $2y + 2z = -14$

   $3(7 + y) + 4y - 2z = -10$

   $21 + 3y + 4y - 2z = -10$

   $7y - 2z = -31$

4. The solution to the system is $x = 1$, $y = -2$, and $z = 3$.