3

Systems Redux

Solving Matrix Equations

Warm Up

Rewrite each expression with the fewest possible terms.

1. $(2x + 7y - 3z) + (-x + 4y + 6z)$
2. $(10x - 4y - 9z) - (3x - 5y + z)$
3. $\frac{1}{2}(-8x + 5y + 12z)$

Learning Goals

- Determine the dimensions of a matrix.
- Identify specific matrix elements.
- Determine the inverse of a matrix.
- Use matrices to solve systems of equations.

Key Terms

- matrix
- dimensions
- square matrix
- matrix element
- matrix multiplication
- identity matrix
- multiplicative inverse of a matrix
- matrix equation
- coefficient matrix
- variable matrix
- constant matrix

You have solved a system of three linear equations algebraically using substitution or Gaussian elimination. How can you use matrices to write and solve a system of linear equations?

GETTING STARTED

Black and Gold Track What Sold

The high school lacrosse team just completed their winter fundraisers. They sold \$20, \$25, and \$50 gasoline cards and \$20, \$25, and \$50 grocery cards during the months of November and December. Two parents volunteered to keep track of the total sales.

Mr. Black's record for the fundraisers is as follows:

November: The team sold 230 gas cards for \$20, 110 gas cards for \$25, and 550 gas cards for \$50. The team sold 112 grocery cards for \$20, 89 grocery cards for \$25, and 95 grocery cards for \$50.

December: The team sold 496 gas cards for \$20, 850 gas cards for \$25, and 348 gas cards for \$50. The team sold 129 grocery cards for \$20, 233 grocery cards for \$25, and 308 grocery cards for \$50.

Ms. Gold's record for the fundraisers is as follows:

$$\begin{array}{l|cc|} & \multicolumn{2}{c}{\text{November}} \\ & \text{Gas} & \text{Grocery} \\ \text{\$20 cards} & 230 & 112 \\ \text{\$25 cards} & 110 & 89 \\ \text{\$50 cards} & 550 & 95 \end{array} + \begin{array}{l|cc|} & \multicolumn{2}{c}{\text{December}} \\ & \text{Gas} & \text{Grocery} \\ \text{\$20 cards} & 496 & 129 \\ \text{\$25 cards} & 850 & 233 \\ \text{\$50 cards} & 348 & 308 \end{array}$$

1. **Do Mr. Black's record and Ms. Gold's record give you the same information?**

2. **Which report is easier to read?**

3. **Calculate the total amount of money collected selling $25 grocery cards.**

4. **Calculate the total amount of money collected selling $50 gas cards.**

5. **Out of the 6 types of cards sold, which card sold the least?**

6. **Which report did you use to answer Questions 3, 4, and 5? Why?**

7. **Is there another way to record the fundraisers using Ms. Gold's method? Explain your reasoning.**

ACTIVITY

3.1 Matrices and Their Inverses

Systems of equations can become cumbersome to solve by hand, particularly when the system contains three or more variables. There is another method that you can use to solve systems of linear equations using technology. It involves a mathematical object called a *matrix*.

Rows are horizontal, while columns are vertical.

A **matrix** (plural **matrices**) is an array of numbers composed of rows and columns. A matrix is usually designated by a capital letter. The **dimensions** of a matrix are its number of rows and its number of columns. A **square matrix** is a matrix that has an equal number of rows and columns.

This matrix is an $n \times m$ matrix, because it has n rows and m columns.

$$\begin{bmatrix} a_{11} & a_{12} \ldots a_{1m} \\ a_{21} & a_{22} \ldots a_{2m} \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad\; \cdot \quad \cdot \\ a_{n1} & a_{n2} \ldots a_{nm} \end{bmatrix}$$

Each number in the matrix is known as a **matrix element**. Each matrix element is labeled using the notation a_{ij}, where a_{ij} means the number in the ith row and the jth column.

$$A = \begin{array}{c} \\ C \\ D \\ E \\ EE \end{array} \begin{array}{c} \begin{array}{ccccc} 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{bmatrix} 3 & 5 & 3 & 5 & 0 \\ 5 & 7 & 6 & 4 & 3 \\ 6 & 5 & 7 & 7 & 2 \\ 2 & 0 & 8 & 5 & 2 \end{bmatrix} \end{array}$$

The Fleet Feet shoe store in your town just got a shipment of Phantoms, a top-selling running shoe. They are available in sizes 9 through 13 and in widths *C*, *D*, *E*, and *EE*. The manager of the store keeps track of the inventory according to the number of pairs of shoes in each size and width using the matrix shown.

1. Consider matrix *A*.

a. Describe what each row and each column represents in terms of the problem situation.

b. What are the dimensions of this matrix?

c. **Determine the number that is in each location. Describe the meaning of the matrix element in terms of the problem situation.**
   - **The element a_{34}**
   - **The element a_{45}**
   - **The element a_{21}**

2. **The Fleet Feet shoe store in a neighboring town carries the same types of shoes and represents its inventory using matrix *B*.**

   a. **What are the dimensions of matrix *B*?**

$$B = \begin{array}{c} \\ C \\ D \\ E \\ EE \end{array} \begin{array}{c} \begin{array}{ccccc} 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{bmatrix} 0 & 4 & 4 & 6 & 1 \\ 1 & 5 & 5 & 7 & 3 \\ 2 & 3 & 3 & 5 & 5 \\ 2 & 2 & 2 & 6 & 4 \end{bmatrix} \end{array}$$

   b. **What matrix element represents the number of size 10E shoes? Write your answer in matrix notation.**

   c. **How is b_{42} different from b_{24}?**

Coach Tirone's team is at their first track meet. The finishing places and number of medals earned for four of her sprinters are displayed in matrix A. Points are awarded according to the scoring matrix B.

$$A = \begin{array}{r} \\ \text{Lauren} \\ \text{Kerri} \\ \text{Meaghan} \\ \text{Erin} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{1st} & \text{2nd} & \text{3rd} & \text{4th} & \text{5th} \end{array} \\ \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \qquad B = \begin{array}{r} \\ \text{1st} \\ \text{2nd} \\ \text{3rd} \\ \text{4th} \\ \text{5th} \end{array} \begin{array}{c} \text{Points} \\ \begin{bmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} \end{array}$$

3. Determine each sprinter's score for the meet.

4. Describe the method you used to calculate each sprinter's score.

5. Record the sprinters' overall scores as a 4 × 1 matrix. Label the rows and columns.

When you determined the matrix representing the overall scores for each sprinter, you actually performed a process called *matrix multiplication*.

In **matrix multiplication**, an element a_{pq} of the product matrix is determined by multiplying each element in row p of the first matrix by an element from column q in the second matrix and calculating the sum of the products. In order to multiply matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix.

Sprinter's Matrix **Scoring Matrix** **Results Matrix**

$$\begin{array}{r} \\ \text{Lauren} \\ \text{Kerri} \\ \text{Meaghan} \\ \text{Erin} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{1st} & \text{2nd} & \text{3rd} & \text{4th} & \text{5th} \end{array} \\ \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \times \begin{array}{r} \\ \text{1st} \\ \text{2nd} \\ \text{3rd} \\ \text{4th} \\ \text{5th} \end{array} \begin{array}{c} \text{Points} \\ \begin{bmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} \end{array} = \begin{array}{r} \\ \text{Lauren} \\ \text{Kerri} \\ \text{Meaghan} \\ \text{Erin} \end{array} \begin{array}{c} \text{Points} \\ \begin{bmatrix} 38 \\ 24 \\ 34 \\ 34 \end{bmatrix} \end{array}$$

6. **Record the dimensions for each of the matrices in the table.**

Dimensions of Sprinter's Matrix		Dimensions of Scoring Matrix		Dimensions of Results Matrixns

Think about:

How does the number of columns in the sprinter's matrix compare to the number rows in the scoring matrix?

7. **Josh made an observation about the dimensions of the product matrix.**

Josh

sprinters by medals × medals by points = sprinters × points

results

Write a generalization about the dimensions of the product matrix, based on Josh's statement.

8. **Consider the matrices** $A = \begin{bmatrix} 1 & 0 & -5 & 3 \\ -2 & 1 & 4 & 7 \end{bmatrix}$ **and** $B = \begin{bmatrix} -3 & 2 \\ 0 & 1 \\ -1 & 5 \\ 2 & 0 \end{bmatrix}$.

a. **Is the product matrix *AB* defined? Explain your reasoning.**

b. **Predict the dimensions of the product matrix *AB*. Explain your reasoning.**

A product matrix *AB* is defined if the number of columns in matrix *A* is the same as the number of rows in matrix *B* so that multiplication can be performed.

The **identity matrix**, *I*, is a square matrix whose elements are 0s and 1s. The 1s are arranged diagonally from upper left to lower right as shown.

$$I_1 = [1]$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Recall that for any real number *n*, the multiplicative inverse is $\frac{1}{n}$ because the product of *n* and $\frac{1}{n}$ is the multiplicative identity, or 1. Similarly, when a matrix is multiplied by its inverse matrix, their product is the *identity matrix*.

The **multiplicative inverse of a matrix** (or just inverse) of *A* is designated as A^{-1}, and is a matrix such that $A \cdot A^{-1} = I$ or the identity matrix. Non-square matrices do not have inverses.

9. RJ used his knowledge of multiplicative inverses to make an assumption about the inverse of matrix *R*. Use matrix multiplication to determine why RJ's conclusion is incorrect.

RJ

$$R = \begin{bmatrix} \frac{2}{3} & 1 \\ 6 & -1 \end{bmatrix}$$

Matrix S is the inverse of matrix R because each matrix element in matrix S is the reciprocal of the corresponding matrix element in matrix R.

$$S = \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{6} & -1 \end{bmatrix}$$

10. Use matrix multiplication to determine whether the two matrices are inverses. Explain your reasoning.

a. $C = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$ $\qquad D = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

b. $G = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ $\qquad H = \begin{bmatrix} 1 & 11 & 14 \\ 0 & -1 & -1 \\ 0 & -3 & -4 \end{bmatrix}$

ACTIVITY
3.2

Solving Systems with Matrices

Matrices can be used to solve a system of equations. A system can be written as a **matrix equation**, or an equation with matrices.

Consider a system of three linear equations in three variables, written in standard form: $Ax + By + Cz = D$. The system can be written as a matrix equation $A \cdot X = B$, by writing it as the product of a *coefficient matrix* and a *variable matrix* equal to a *constant matrix*.

A **coefficient matrix** is a square matrix that consists of each coefficient of each equation in the system of equations, in order, when they are written in standard form. A **variable matrix** is a matrix in one column that represents all of the variables in the system of equations. A **constant matrix** is a matrix in one column that represents each of the constants in the system of equations.

In a system with n equations, the coefficient matrix will be an $n \times n$ matrix, the variable matrix will be an $n \times 1$ matrix, and the constant matrix will be an n x 1 matrix.

Worked Example

Consider the system of three linear equations in three variables:

$$\begin{cases} 2x - y - 2z = 3 \\ 3x + y - 2z = 11 \\ -2x - y + z = -8. \end{cases}$$

Each equation is written in standard form $Ax + By + Cz = D$.

In the matrix equation $A \cdot X = B$, A is a 3×3 matrix consisting of each coefficient, X is a 3×1 matrix representing all of the variables in the system, and B is a 3×1 matrix consisting of each constant term.

$$\begin{bmatrix} 2 & -1 & -2 \\ 3 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix}$$

$$\begin{tikzcd} \text{Coefficient Matrix} \arrow[r] & A & \cdot & X & = & B & \text{Constant Matrix} \arrow[l] \\ & & & \text{Variable Matrix} \arrow[u] & & & \end{tikzcd}$$

1. **Verify that the matrix equation is an equivalent representation of the system of equations.**

   a. **Use matrix multiplication to calculate $A \cdot X$.**

   b. **Write the system of equations that the matrix equation represents. Justify your reasoning.**

   c. **Is the matrix equation an equivalent representation of the system of equations?**

2. **Write each system of equations as a matrix equation.**

   a. $\begin{cases} x + 2y - 2z = -8 \\ 2x - y + z = -1 \\ -x + 3y + 2z = 13 \end{cases}$

   b. $\begin{cases} 3x + 2y + 2z = 4 \\ 5x = -4y \\ -x + 2y - z = 12 \end{cases}$

3. **Describe the error in Antoine's method.**

Think about:

Is each equation of the form $Ax + By + Cz = D$?

Antoine

$$\begin{cases} 2x - 4y = 5z - 1 \\ -x = 5y - 3z \\ 3x - 5y - 3z = 14 \end{cases}$$

$$\begin{bmatrix} 2 & -4 & 5 \\ -1 & 2 & -3 \\ 3 & -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 14 \end{bmatrix}$$

Remember, you can use technology with matrices as a tool when solving matrix equations.

Worked Example

Consider the system of equations $\begin{cases} 2x - y - 2z = 3 \\ 3x + y - 2z = 11 \\ -2x - y + z = -8 \end{cases}$

Step 1: Write the system as a matrix equation.

$$\underset{A}{\begin{bmatrix} 2 & -1 & -2 \\ 3 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix}}.$$

Step 2: Calculate the inverse of A using yourchoice of technolgy.

$$A^{-1} = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 2 & 2 \\ 1 & -4 & -5 \end{bmatrix}$$

Step 3: Use technology to multiply the inverse by matrix B to calculate the solution to the system.

$$\begin{bmatrix} 1 & -3 & -4 \\ -1 & 2 & 2 \\ 1 & -4 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

The solution to the matrix equation is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, or (2, 3, −1).

4. Anthony is asked to solve a system of three linear equations using technology with matrices and he gets an error message letting him know it is a noninvertible matrix.

Anthony

$$\begin{cases} x + 2y - z = 1 \\ 2x - 2y - z = 2 \\ 3x + 6y - 3z = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & -1 \\ 3 & 6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A \cdot X = B$$

$$X = A^{-1} \cdot B$$

Why might Anthony have seen the error message?

Recall that when any row of a matrix is a multiple of another row, or any column is a multiple of another column, the inverse of the matrix does not exist. When the coefficient matrix is noninvertible, the system is not able to be solved. In this case, the system will either have many solutions or no solution.

When using technology, you will get a similar error message for both cases, because in both cases, the inverse of the coefficient matrix does not exist. You can use the constant matrix to differentiate between a system with many solutions and a system with no solution.

Worked Example

Many Solutions

$$\begin{cases} x + 5y - 2z = 3 \\ 2x + 10y - 4z = 6 \\ x + y + 3z = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & -2 \\ 2 & 10 & -4 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$$

If each variable in one equation is a multiple of another equation, and the constants are a multiple by the same factor, the system will have many solutions.

No Solution

$$\begin{cases} x + 5y - 2z = 3 \\ 2x + 10y - 4z = 7 \\ x + y + 3z = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & -2 \\ 2 & 10 & -4 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}$$

If each variable in one equation is a multiple of another equation, and the constants are not a multiple by the same factor, the system will have no solution.

5. Determine whether the system of equations in Question 4 has no solution or many solutions.

6. **Two of the equations in the system of three linear equations are given.**

$$\begin{cases} 5x - 3y + z = -4 \\ x + 2y - 3z = 0 \end{cases}$$

   a. **Write a third equation that would produce a system with no solution. Explain your reasoning.**

   b. **Write a third equation that would produce a system with many solutions.**

7. **Write each system of equations as a matrix equation. Then calculate the solution to each system of linear equations by using technology with matrices.**

Don't forget to fill the variables that have a coefficient of zero.

   a. $$\begin{cases} 2x - 3y = 7 \\ y + z = -5 \\ x + 2y + 4z = -17 \end{cases}$$

   b. $$\begin{cases} 5x + y + 3z = 9 \\ -x - 2y - z = -16 \\ 2x + 4y + 2z = -30 \end{cases}$$

   c. $$\begin{cases} x - 4y + 3z = -7 \\ 2x + 3y - 5z = 19 \\ 4x + y - z = 17 \end{cases}$$

   d. $$\begin{cases} 2x - 3z = 4 \\ 2x + y - 5z = -1 \\ 3y - 4z = 2 \end{cases}$$

NOTES

TALK the TALK

Show Us Your Stuff

As you learned in a previous lesson, you can model a real-world situation using a system of linear equations. Now you have another tool to use to help you solve the system.

1. **Your school's Key Club decided to sell fruit baskets to raise money for a local charity. The club sold a total of 80 fruit baskets. There were three different types of fruit baskets. Small fruit baskets sold for $15.75 each, medium fruit baskets sold for $25 each, and large fruit baskets sold for $32.50 each. The Key Club took in a total of $2086.25, and they sold twice as many large baskets as small baskets.**

   a. **Formulate a system of three linear equations in three variables to represent this problem situation. Be sure to define your variables.**

   b. **Solve the system of three linear equations using technology with matrices. Write your answer in terms of the problem situation.**

   c. **Describe the solution in terms of the problem situation.**

2. **Is it always easier to solve a system of linear equations using matrices? Explain your reasoning.**