3

Systems Redux

Solving Matrix Equations

MATERIALS

Technology that can operate with matrices

Lesson Overview

Students are introduced to identity and inverse matrices. They express a system of equation as a matrix eqution. Students relate solving a matrix equation to solving a linear equation, and then use technology to solve a matrix equation. As a culminating activity, they model a scenario with a system of equations, convert it to a matrix equation, solve the matrix equation using technology, and interpret the solution in terms of the scenario.

Algebra 2
Systems of Equations and Inequalities

(3) The student applies mathematical processes to formulate systems of equations and inequalities, uses a variety of methods to solve, and analyzes reasonableness of solutions. The student is expected to:

(B) solve systems of three linear equations in three variables by using Gaussian elimination, technology with matrices, and substitution.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I 3.D, 3.E, 4.B, 4.C, 5.B, 5.F, 5.G

Essential Ideas

- The multiplicative identity matrix, I, is a square matrix such that for any matrix A, $A \cdot I = A$.
- The multiplicative inverse of a matrix of a square matrix A is designated as A^{-1}, and is a matrix such that $A \cdot A^{-1} = I$.
- Matrices can be used to solve a system of equations in the form $Ax + By + Cz = D$ by writing the system as a matrix equation in the form $A \cdot X = B$, where A represents the coefficient matrix, X represents the variable matrix, and B represents the constant matrix.
- Technology can be used to solve matrix equations.

Lesson Structure and Pacing: 2 Days

Day 1

Engage

Getting Started: Black and Gold Track What Sold

Students analyze two different representations of information about a fundraiser. One representation uses running text to list the number of items sold in two different months and the other representation presents the same information in matrix form. They respond to questions by referencing the data and comparing the representations.

Develop

Activity 3.1: Matrices and Their Inverses

The terms *matrix, dimensions, square matrix, matrix element, matrix multiplication, identity matrix*, and *multiplicative inverse* are defined. Using a real-world context, students explore the components of a matrix. They multiply matrices and use matrix multiplication to determine whether two matrices are inverses.

Day 2

Activity 3.2: Solving Systems with Matrices

Students are introduced to matrices as a way to represent and solve systems of equations. They analyze a series of worked examples that demonstrate how to express a system of equations as a matrix equation, how to use technology to solve matrix equations, and how to recognize whether a matrix equation has many solutions or no solutions. Students conclude the activity by solving systems of equations with matrices.

Demonstrate

Talk the Talk: Show Us Your Stuff

Students encounter a scenario that can be represented using a system of equations. They write a system of equations representing the situation, express the system as a matrix equation, and then use technology with matrices to solve the matrix equation. Students also interpret their solution in the context of the problem situation.

Getting Started: Black and Gold Track What Sold

ENGAGE

Facilitation Notes

In this activity, students analyze two different representations of information about a fundraiser. One representation uses running text to list the number of items sold in two different months and the other representation presents the same information in matrix form. They respond to questions by referencing the data and comparing the representations.

Have students work with a partner or in a group to read the problems and answer Questions 1 through 7. Share responses as a class.

Differentiation strategy

To extend the activity, provide students the first paragraph and Mr. Black's records, but do not show them Ms. Gold's records. Ask them to organize the information, and then compare their strategies with Ms. Gold's organization.

Questions to ask

- What are the advantages to Ms. Gold's organization?
- How did you identify the data required to respond to this question?
- What order of operations did you use to answer Question 3?
- Is there another order that could have been used to answer Question 3? If so, explain the process.
- What is another question you could answer by referring to the data?

Summary

Data organized in rows and columns allows for easier identification and interpretation of the information.

Activity 3.1

Matrices and Their Inverses

DEVELOP

Facilitation Notes

In this activity, the terms *matrix, dimensions, square matrix, matrix element, matrix multiplication, identity matrix,* and *multiplicative inverse* are defined. Using a real-world context, students explore the components of a matrix. They multiply matrices and use matrix multiplication to determine whether two matrices are inverses.

Have students work with a partner or in a group to read the introduction and complete Questions 1 and 2. Share responses as a class.

Differentiation strategy

To scaffold support, remind students that they used rectangular arrays to model multiplication problems in elementary school. They also labeled these arrays using the notation *row* × *column*.

Questions to ask

- What is a matrix?
- How is a matrix like a table? How is a matrix different than a table?
- When identifying the dimensions of a matrix or the location of an element in a matrix, is the column or row expressed first?
- How is a 2 × 5 matrix different than a 5 × 2 matrix?
- What does the 0 in matrix *B* represent?

Have students work with a partner or in a group to complete Questions 3 through 5. Share responses as a class.

Questions to ask

- Explain why it is necessary to have five values in the second matrix.
- Explain the relationship between a row in the first matrix and the column in the second matrix.
- Why does the resulting matrix only have four elements?
- Why does it make sense that the resulting matrix has four rows?

Ask a student to read the information following Question 5. Discuss as a class.

Questions to ask

- How does the a_{pq} notation relate to the example?
- Explain how each element in the results matrix is the result of both multiplication and addition.

Have students work with a partner or in a group to complete Questions 6 through 8. Share responses as a class.

Questions to ask

- How can you tell by the dimensions whether or not you can multiply two matrices?
- Provide an example of the dimensions of two matrices that can be multiplied.

Differentiation strategy

To scaffold support for all students, provide directions on how to use technology to multiply matrices.

Have students work with a partner or in a group to complete Questions 9 and 10. Share responses as a class.

As students work, look for

Whether they use mental math or write out all the operations when calculating the value of each element in the result matrix.

Questions to ask

- If the product matrix $A \cdot A^{-1}$ does not equal the identity matrix I, what does this tell you about the two matrices?
- How did you calculate each element in the product matrix?

Summary

A matrix is an array of numbers, known as matrix elements, composed into rows and columns. You can determine an element a_{pq} of the product matrix by multiplying each element in row p of the first matrix by an element from column q in the second matrix and calculating the sum of the products. The product of a matrix and its multiplicative inverse is the identity matrix.

Activity 3.2

Solving Systems with Matrices

Facilitation Notes

In this activity, students are introduced to matrices as a way to represent and solve systems of equations. They analyze a series of Worked Examples that demonstrate how to express a system of equations as a matrix equation, how to use technology to solve matrix equations, and how to recognize whether a matrix equation has many solutions or no solutions. Students conclude the activity by solving systems of equations with matrices.

Ask a student to read the introduction and analyze the Worked Example as a class. Have students work with a partner or in a group to complete Question 1.

Misconception

Students may be confused by the equation $A \cdot X = B$ thinking it is referring to the same variables as in the referenced equation $Ax + By + Cz = D$. Clarify this misunderstanding by explaining that A, X, and B represent matrices, while A, B, C and D represent constants, and x, y, and z represent individual variables.

Questions to ask

- How is a coefficient matrix formed?
- Why does the X matrix have three elements?
- Why do you think X is a 3×1 matrix?

- Use the dimensions of A and X to explain why B has the dimensions it has.

Have students work with a partner or in a group to complete Questions 2 and 3. Share responses as a class.

Questions to ask

- Explain how the matrix equation $A \cdot X = B$ is another way to represent the system of equations.
- How do you account for missing variables in a system of equations when you represent the system as a matrix equation?
- How would you correct Antoine's work?

Have students work with a partner or in a group to complete Question 4. Share responses as a class.

Questions to ask

- How is a coefficient matrix formed?
- Why does the X matrix have three elements?
- Why do you think X is a 3×1 matrix?
- Use the dimensions of A and X to explain why B has the dimensions it has.

Have students read the Worked Example and then work with a partner or in a group to complete Questions 4. Share responses as a class.

Questions to ask

- According to the Worked Example, what additional step can the technology do for you when solving a matrix equation?
- How does this error message relate to the system of equations?

Have students read the worked example following Question 4 and complete Questions 5 through 6.

Differentiation strategy

To scaffold support, provide examples of systems of equations in two variables for them to solve using the linear combination method to help them relate to the concepts of many solutions and no solutions in the worked example.

$x + 2y = 8$	$x + 2y = 8$
$3x + 6y = 24$	$3x + 6y = 23$

Questions to ask

- How can you tell when a matrix equation has many solutions or no solutions?
- Can you identify whether a matrix equation has many solutions or no solutions without the use of technology? Explain.
- How did you build an equation to get the results you wanted?

Have students work with a partner or in a group to complete Question 7.

Differentiation strategy

To scaffold support for all students, provide directions to use technology to solve a matrix equation.

Consider the system of equations $\begin{cases} 2x - y - 2z = 3 \\ 3x + y - 2z = 11 \\ -2x - y + z = -8 \end{cases}$.

Step 1: Write the system as a matrix equation.

$$\begin{array}{ccccc} \begin{bmatrix} 2 & -1 & -2 \\ 3 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix} & \begin{bmatrix} x \\ y \\ z \end{bmatrix} & = & \begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix} \\ A & \cdot \; X & = & B \end{array}$$

Step 2: Calculate the inverse of A.

Step 3: Multiply the inverse by matrix B to calculate the solution to the system.

Ans*[B]

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

The solution to the matrix equation is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -11 \end{bmatrix}$, or (2, 3, −1).

Questions to ask

- Explain your steps to solve the system of equations.
- How can you check your solution?

Summary

Matrices can be used to solve a system of equations in the form $Ax + By + Cz = D$ by writing the system as a matrix equation in the form $A \cdot X = B$, where A represents the coefficient matrix, X represents the variable matrix, and B represents the constant matrix. The matrix equation can then be solved using technology.

DEMONSTRATE

Talk the Talk: Show Us Your Stuff

Facilitation Notes

In this activity, students encounter a scenario that can be represented using a system of equations. They write a system of equations representing the situation, expresss the system as a matrix equation, and then use technology with matrices to solve the matrix equation. Students also interpret their solution in the context of the problem situation.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Questions to ask

- Explain how your system of equations models the context.
- How did you rewrite your system of equations using matrices?
- Explain how you used technology to solve your matrix equation.

Summary

A real-world situation that can be modeled by a system of linear equations may be solved using a matrix equation and technology.

3

Systems Redux

Solving Matrix Equations

Warm Up

Rewrite each expression with the fewest possible terms.

1. $(2x + 7y - 3z) + (-x + 4y + 6z)$
2. $(10x - 4y - 9z) - (3x - 5y + z)$
3. $\frac{1}{2}(-8x + 5y + 12z)$

Learning Goals

- Determine the dimensions of a matrix.
- Identify specific matrix elements.
- Determine the inverse of a matrix.
- Use matrices to solve systems of equations.

Key Terms

- matrix
- dimensions
- square matrix
- matrix element
- matrix multiplication
- identity matrix
- multiplicative inverse of a matrix
- matrix equation
- coefficient matrix
- variable matrix
- constant matrix

You have solved a system of three linear equations algebraically using substitution or Gaussian elimination. How can you use matrices to write and solve a system of linear equations?

Warm Up Answers

1. $x + 11y + 3z$
2. $7x + y - 10z$
3. $-4x + \frac{5}{2}y + 6z$

Answers

1. Yes, both reports contain the same information.
2. Sample answer. Ms. Gold's report is more straightforward, concise, and easier to read.

GETTING STARTED

Black and Gold Track What Sold

The high school lacrosse team just completed their winter fundraisers. They sold \$20, \$25, and \$50 gasoline cards and \$20, \$25, and \$50 grocery cards during the months of November and December. Two parents volunteered to keep track of the total sales.

Mr. Black's record for the fundraisers is as follows:

November: The team sold 230 gas cards for \$20, 110 gas cards for \$25, and 550 gas cards for \$50. The team sold 112 grocery cards for \$20, 89 grocery cards for \$25, and 95 grocery cards for \$50.

December: The team sold 496 gas cards for \$20, 850 gas cards for \$25, and 348 gas cards for \$50. The team sold 129 grocery cards for \$20, 233 grocery cards for \$25, and 308 grocery cards for \$50.

Ms. Gold's record for the fundraisers is as follows:

$$\begin{array}{l} \\ \\ \text{\$20 cards} \\ \text{\$25 cards} \\ \text{\$50 cards} \end{array} \begin{array}{c} \text{November} \\ \begin{array}{cc} \text{Gas} & \text{Grocery} \end{array} \\ \begin{bmatrix} 230 & 112 \\ 110 & 89 \\ 550 & 95 \end{bmatrix} \end{array} + \begin{array}{l} \\ \\ \text{\$20 cards} \\ \text{\$25 cards} \\ \text{\$50 cards} \end{array} \begin{array}{c} \text{December} \\ \begin{array}{cc} \text{Gas} & \text{Grocery} \end{array} \\ \begin{bmatrix} 496 & 129 \\ 850 & 233 \\ 348 & 308 \end{bmatrix} \end{array}$$

1. Do Mr. Black's record and Ms. Gold's record give you the same information?

2. Which report is easier to read?

3. Calculate the total amount of money collected selling $25 grocery cards.

4. Calculate the total amount of money collected selling $50 gas cards.

5. Out of the 6 types of cards sold, which card sold the least?

6. Which report did you use to answer Questions 3, 4, and 5? Why?

7. Is there another way to record the fundraisers using Ms. Gold's method? Explain your reasoning.

3. (89 + 233)25 = $8050
   $8050 was collected selling $25 grocery cards.
4. (550 + 348)50 = $44, 900
   $44,900 was collected selling $50 gas cards.
5. The $25 grocery card sold the least.
6. Sample answer.
   I used Ms. Gold's report to answer Questions 3, 4, and 5 because it was easier to read.
7. Sample answer.
   Yes. the amount of the gift cards could have been listed at the top of three columns and the type of gift card could have been listed beside each of two rows.

Answers

1a. Each row represents the number of pairs of shoes of a given width. Each column represents the number of pairs of shoes of a given size.

1b. The dimensions of the matrix are 4 × 5.

ACTIVITY **3.1**

Matrices and Their Inverses

Systems of equations can become cumbersome to solve by hand, particularly when the system contains three or more variables. There is another method that you can use to solve systems of linear equations using technology. It involves a mathematical object called a *matrix*.

Rows are horizontal, while columns are vertical.

A **matrix** (plural **matrices**) is an array of numbers composed of rows and columns. A matrix is usually designated by a capital letter. The **dimensions** of a matrix are its number of rows and its number of columns. A **square matrix** is a matrix that has an equal number of rows and columns.

This matrix is an $n \times m$ matrix, because it has n rows and m columns.

$$\begin{bmatrix} a_{11} & a_{12} \ldots a_{1m} \\ a_{21} & a_{22} \ldots a_{2m} \\ . & . \quad\quad . \\ . & . \quad . \quad . \\ . & . \quad . \quad . \\ a_{n1} & a_{n2} \ldots a_{nm} \end{bmatrix}$$

Each number in the matrix is known as a **matrix element**. Each matrix element is labeled using the notation a_{ij}, where a_{ij} means the number in the ith row and the jth column.

$$A = \begin{array}{c} \\ C \\ D \\ E \\ EE \end{array} \begin{array}{c} \begin{array}{ccccc} 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{bmatrix} 3 & 5 & 3 & 5 & 0 \\ 5 & 7 & 6 & 4 & 3 \\ 6 & 5 & 7 & 7 & 2 \\ 2 & 0 & 8 & 5 & 2 \end{bmatrix} \end{array}$$

The Fleet Feet shoe store in your town just got a shipment of Phantoms, a top-selling running shoe. They are available in sizes 9 through 13 and in widths *C*, *D*, *E*, and *EE*. The manager of the store keeps track of the inventory according to the number of pairs of shoes in each size and width using the matrix shown.

1. Consider matrix *A*.

a. Describe what each row and each column represents in terms of the problem situation.

b. What are the dimensions of this matrix?

ELL Tip

For students who confuse the orientation of rows and columns, have them connect the meaning of *column* to the vertical columns of some buildings.

Help students understand what a matrix is by relating it to a table.

A matrix is a table that has both row and column titles, although the lines separating the cells are removed.

c. **Determine the number that is in each location. Describe the meaning of the matrix element in terms of the problem situation.**
   - **The element a_{34}**
   - **The element a_{45}**
   - **The element a_{21}**

2. **The Fleet Feet shoe store in a neighboring town carries the same types of shoes and represents its inventory using matrix *B*.**

   a. **What are the dimensions of matrix *B*?**

$$B = \begin{array}{c} \\ C \\ D \\ E \\ EE \end{array} \begin{array}{c} \begin{array}{ccccc} 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{bmatrix} 0 & 4 & 4 & 6 & 1 \\ 1 & 5 & 5 & 7 & 3 \\ 2 & 3 & 3 & 5 & 5 \\ 2 & 2 & 2 & 6 & 4 \end{bmatrix} \end{array}$$

   b. **What matrix element represents the number of size 10E shoes? Write your answer in matrix notation.**

   c. **How is b_{42} different from b_{24}?**

Answers

1c. • The element a_{34} represents the number of size 12*E* shoes. There are 7 pairs.

• The element a_{45} represents the number of size 13*EE* shoes. There are 2 pairs.

• The element a_{21} represents the number of size 9*D* shoes. There are 5 pairs.

2a. The dimensions of matrix *B* are 4×5.

2b. The number of size 10*E* shoes are located in the 3rd row, 2nd column of matrix *B*, or matrix element b_{32}.

2c. Size 10*EE* shoes are represented by b_{42}, which is the fourth row, second column. Size 12*D* shoes are represented by b_{24}, which is the second row, fourth column.

Answers

3. Lauren:
   3(10) + 1(8) + 0(6) + 0(4) + 0(2)
   = 30 + 8 + 0 + 0 + 0 = 38 points

   Kerri:
   0(10) + 2(8) + 0(6) + 1(4) + 2(2)
   = 0 + 16 + 0 + 4 + 4 = 24 points

   Meaghan:
   0(10) + 1(8) + 3(6) + 1(4) + 2(2)
   = 0 + 8 + 18 + 4 + 4 = 34 points

   Erin:
   2(10) + 1(8) + 0(6) + 1(4) + 1(2)
   = 20 + 8 + 0 + 4 + 2 = 34 points

4. For each sprinter, I multiplied the assigned point value by the number of medals they earned at each place, and then added all of their points to get each sprinter's overall score.

5. $$\begin{array}{r c} & \text{Points} \\ \begin{array}{r} \text{Lauren} \\ \text{Kerri} \\ \text{Meaghan} \\ \text{Erin} \end{array} & \begin{bmatrix} 38 \\ 24 \\ 34 \\ 34 \end{bmatrix} \end{array}$$

Coach Tirone's team is at their first track meet. The finishing places and number of medals earned for four of her sprinters are displayed in matrix A. Points are awarded according to the scoring matrix B.

$$A = \begin{array}{r c} & \begin{array}{ccccc} \text{1st} & \text{2nd} & \text{3rd} & \text{4th} & \text{5th} \end{array} \\ \begin{array}{r} \text{Lauren} \\ \text{Kerri} \\ \text{Meaghan} \\ \text{Erin} \end{array} & \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \qquad B = \begin{array}{r c} & \text{Points} \\ \begin{array}{r} \text{1st} \\ \text{2nd} \\ \text{3rd} \\ \text{4th} \\ \text{5th} \end{array} & \begin{bmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} \end{array}$$

3. Determine each sprinter's score for the meet.

4. Describe the method you used to calculate each sprinter's score.

5. Record the sprinters' overall scores as a 4 × 1 matrix. Label the rows and columns.

When you determined the matrix representing the overall scores for each sprinter, you actually performed a process called *matrix multiplication*.

In **matrix multiplication**, an element a_{pq} of the product matrix is determined by multiplying each element in row p of the first matrix by an element from column q in the second matrix and calculating the sum of the products. In order to multiply matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix.

Sprinter's Matrix **Scoring Matrix** **Results Matrix**

$$\begin{array}{r c} & \begin{array}{ccccc} \text{1st} & \text{2nd} & \text{3rd} & \text{4th} & \text{5th} \end{array} \\ \begin{array}{r} \text{Lauren} \\ \text{Kerri} \\ \text{Meaghan} \\ \text{Erin} \end{array} & \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \times \begin{array}{r c} & \text{Points} \\ \begin{array}{r} \text{1st} \\ \text{2nd} \\ \text{3rd} \\ \text{4th} \\ \text{5th} \end{array} & \begin{bmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} \end{array} = \begin{array}{r c} & \text{Points} \\ \begin{array}{r} \text{Lauren} \\ \text{Kerri} \\ \text{Meaghan} \\ \text{Erin} \end{array} & \begin{bmatrix} 38 \\ 24 \\ 34 \\ 34 \end{bmatrix} \end{array}$$

6. Record the dimensions for each of the matrices in the table.

Dimensions of Sprinter's Matrix		Dimensions of Scoring Matrix		Dimensions of Results Matrixns

Think about:

How does the number of columns in the sprinter's matrix compare to the number rows in the scoring matrix?

7. Josh made an observation about the dimensions of the product matrix.

Josh

$\underbrace{\underline{\text{sprinters}} \text{ by medals} \times \text{medals by } \underline{\text{points}}}_{\text{results}} = \underline{\text{sprinters}} \times \underline{\text{points}}$

Write a generalization about the dimensions of the product matrix, based on Josh's statement.

8. Consider the matrices $A = \begin{bmatrix} 1 & 0 & -5 & 3 \\ -2 & 1 & 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ 0 & 1 \\ -1 & 5 \\ 2 & 0 \end{bmatrix}$.

a. Is the product matrix *AB* defined? Explain your reasoning.

b. Predict the dimensions of the product matrix *AB*. Explain your reasoning.

A product matrix *AB* is defined if the number of columns in matrix *A* is the same as the number of rows in matrix *B* so that multiplication can be performed.

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6.

Dimensions of Sprinter's Matrix		Dimensions of Scoring Matrix		Dimensions of Results Matrixns
4×5		5×1		4×1

Answers

6. See table below.
7. When the product of two matrices is defined, the dimensions of the product matrix are the "outer dimensions," or the number of rows of the first matrix by the number of columns of the second matrix.

8a. Yes. The product matrix *AB* is defined. Matrix *A* has 4 columns and matrix *B* has 4 rows.

8b. The product matrix is a 2×2 matrix. The "outer dimensions" of the multiplied matrices are 2 and 2.

$2 \times 4 \times 4 \times 2 \rightarrow 2 \times 2$

Answers

9. RJ's reasoning is incorrect because when matrix R is multiplied by matrix S, it does not produce the identity matrix.

$$\begin{bmatrix} \frac{2}{3} & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{6} & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{7}{6} & -\frac{1}{3} \\ \frac{53}{6} & 7 \end{bmatrix}$$

10a. Yes. Matrix C and matrix D are inverses because their product matrix CD yields the multiplicative identity matrix.

$$\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

10b. No. Matrix G and matrix H are not inverses because their product matrix GH does not yield the multiplicative identity matrix.

$$GH = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & -3 \end{bmatrix}$$

The **identity matrix**, I, is a square matrix whose elements are 0s and 1s. The 1s are arranged diagonally from upper left to lower right as shown.

$I_1 = [1]$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Recall that for any real number n, the multiplicative inverse is $\frac{1}{n}$ because the product of n and $\frac{1}{n}$ is the multiplicative identity, or 1. Similarly, when a matrix is multiplied by its inverse matrix, their product is the *identity matrix*.

The **multiplicative inverse of a matrix** (or just inverse) of A is designated as A^{-1}, and is a matrix such that $A \cdot A^{-1} = I$ or the identity matrix. Non-square matrices do not have inverses.

9. RJ used his knowledge of multiplicative inverses to make an assumption about the inverse of matrix *R*. Use matrix multiplication to determine why RJ's conclusion is incorrect.

RJ

$R = \begin{bmatrix} \frac{2}{3} & 1 \\ 6 & -1 \end{bmatrix}$

Matrix S is the inverse of matrix R because each matrix element in matrix S is the reciprocal of the corresponding matrix element in matrix R.

$S = \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{6} & -1 \end{bmatrix}$

10. Use matrix multiplication to determine whether the two matrices are inverses. Explain your reasoning.

a. $C = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$ $\qquad D = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

b. $G = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ $\qquad H = \begin{bmatrix} 1 & 11 & 14 \\ 0 & -1 & -1 \\ 0 & -3 & -4 \end{bmatrix}$

ACTIVITY 3.2

Solving Systems with Matrices

Matrices can be used to solve a system of equations. A system can be written as a **matrix equation**, or an equation with matrices.

Consider a system of three linear equations in three variables, written in standard form: $Ax + By + Cz = D$. The system can be written as a matrix equation $A \cdot X = B$, by writing it as the product of a *coefficient matrix* and a *variable matrix* equal to a *constant matrix*.

A **coefficient matrix** is a square matrix that consists of each coefficient of each equation in the system of equations, in order, when they are written in standard form. A **variable matrix** is a matrix in one column that represents all of the variables in the system of equations. A **constant matrix** is a matrix in one column that represents each of the constants in the system of equations.

In a system with n equations, the coefficient matrix will be an $n \times n$ matrix, the variable matrix will be an $n \times 1$ matrix, and the constant matrix will be an n x 1 matrix.

Worked Example

Consider the system of three linear equations in three variables:

$$\begin{cases} 2x - y - 2z = 3 \\ 3x + y - 2z = 11 \\ -2x - y + z = -8. \end{cases}$$

Each equation is written in standard form $Ax + By + Cz = D$.

In the matrix equation $A \cdot X = B$, A is a 3×3 matrix consisting of each coefficient, X is a 3×1 matrix representing all of the variables in the system, and B is a 3×1 matrix consisting of each constant term.

$$\begin{bmatrix} 2 & -1 & -2 \\ 3 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix}$$

$$\text{Coefficient Matrix} \longrightarrow A \cdot \underset{\substack{\uparrow \\ \text{Variable Matrix}}}{X} = B \longleftarrow \text{Constant Matrix}$$

Answers

1a. $\begin{bmatrix} 2 & -1 & -2 \\ 3 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix}$

$\begin{bmatrix} 2x - 1y - 2z \\ 3x + 1y - 2z \\ -2x - 1y + 1z \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix}$

1b. $\begin{cases} 2x - y - 2z = 3 \\ 3x + y - 2z = 11 \\ -2x - y + z = -8 \end{cases}$

If matrices are equal then corresponding elements are equal.

1c. Yes. The matrix equation is an equivalent representation of the system of equations.

2a. $\begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & 1 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ -1 \\ 13 \end{bmatrix}$

2b. $\begin{cases} 3x + 2y + 2z = 4 \\ 5x + 4y \quad\quad = 0 \\ -x + 2y - z \;= 12 \end{cases}$

$\begin{bmatrix} 3 & 2 & 2 \\ 5 & 4 & 0 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix}$

3. Antoine did not write the system in standard form $Ax + By + Cz = D$ before writing it in matrix form.

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1. Verify that the matrix equation is an equivalent representation of the system of equations.

a. Use matrix multiplication to calculate $A \cdot X$.

b. Write the system of equations that the matrix equation represents. Justify your reasoning.

c. Is the matrix equation an equivalent representation of the system of equations?

2. Write each system of equations as a matrix equation.

a. $\begin{cases} x + 2y - 2z \;= -8 \\ 2x - y + z \;\;= -1 \\ -x + 3y + 2z = 13 \end{cases}$

b. $\begin{cases} 3x + 2y + 2z = 4 \\ 5x = -4y \\ -x + 2y - z = 12 \end{cases}$

3. Describe the error in Antoine's method.

Think about:

Is each equation of the form $Ax + By + Cz = D$?

Antoine

$\begin{cases} 2x - 4y = 5z - 1 \\ -x = 5y - 3z \\ 3x - 5y - 3z = 14 \end{cases}$

$\begin{bmatrix} 2 & -4 & 5 \\ -1 & 2 & -3 \\ 3 & -5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 14 \end{bmatrix}$

Remember, you can use technology with matrices as a tool when solving matrix equations.

Worked Example

Consider the system of equations $\begin{cases} 2x - y - 2z = 3 \\ 3x + y - 2z = 11 \\ -2x - y + z = -8 \end{cases}$

Step 1: Write the system as a matrix equation.

$$\underset{A}{\begin{bmatrix} 2 & -1 & -2 \\ 3 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix}}.$$

Step 2: Calculate the inverse of A using yourchoice of technolgy.

$$A^{-1} = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 2 & 2 \\ 1 & -4 & -5 \end{bmatrix}$$

Step 3: Use technology to multiply the inverse by matrix B to calculate the solution to the system.

$$\begin{bmatrix} 1 & -3 & -4 \\ -1 & 2 & 2 \\ 1 & -4 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ 11 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

The solution to the matrix equation is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, or $(2, 3, -1)$.

4. Anthony is asked to solve a system of three linear equations using technology with matrices and he gets an error message letting him know it is a noninvertible matrix.

Anthony

$$\begin{cases} x + 2y - z = 1 \\ 2x - 2y - z = 2 \\ 3x + 6y - 3z = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & -1 \\ 3 & 6 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A \cdot X = B$$
$$X = A^{-1} \cdot B$$

Why might Anthony have seen the error message?

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Answer

4. In the coefficient matrix of this system of three linear equations, the third row is a multiple of 3 of the first row. When any row of a matrix is a multiple of another row, or any column is a multiple of another column, the inverse does not exist. Therefore, the system cannot be solved.

Answers

5. The system of equations in Question 4 has many solutions.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & -1 \\ 3 & 6 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The third row of the coefficient matrix is a multiple of the first row in the coefficient matrix by a factor 3. The third row in the constant matrix is also a multiple of the first row in the constant matrix by a factor 3. Therefore, the system has many solutions.

Recall that when any row of a matrix is a multiple of another row, or any column is a multiple of another column, the inverse of the matrix does not exist. When the coefficient matrix is noninvertible, the system is not able to be solved. In this case, the system will either have many solutions or no solution.

When using technology, you will get a similar error message for both cases, because in both cases, the inverse of the coefficient matrix does not exist. You can use the constant matrix to differentiate between a system with many solutions and a system with no solution.

Worked Example

Many Solutions

$$\begin{cases} x + 5y - 2z = 3 \\ 2x + 10y - 4z = 6 \\ x + y + 3z = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & -2 \\ 2 & 10 & -4 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$$

If each variable in one equation is a multiple of another equation, and the constants are a multiple by the same factor, the system will have many solutions.

No Solution

$$\begin{cases} x + 5y - 2z = 3 \\ 2x + 10y - 4z = 7 \\ x + y + 3z = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & -2 \\ 2 & 10 & -4 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}$$

If each variable in one equation is a multiple of another equation, and the constants are not a multiple by the same factor, the system will have no solution.

5. Determine whether the system of equations in Question 4 has no solution or many solutions.

6. **Two of the equations in the system of three linear equations are given.**

$$\begin{cases} 5x - 3y + z = -4 \\ x + 2y - 3z = 0 \end{cases}$$

a. **Write a third equation that would produce a system with no solution. Explain your reasoning.**

b. **Write a third equation that would produce a system with many solutions.**

7. **Write each system of equations as a matrix equation. Then calculate the solution to each system of linear equations by using technology with matrices.**

Don't forget to fill the variables that have a coefficient of zero.

a. $$\begin{cases} 2x - 3y = 7 \\ y + z = -5 \\ x + 2y + 4z = -17 \end{cases}$$

b. $$\begin{cases} 5x + y + 3z = 9 \\ -x - 2y - z = -16 \\ 2x + 4y + 2z = -30 \end{cases}$$

c. $$\begin{cases} x - 4y + 3z = -7 \\ 2x + 3y - 5z = 19 \\ 4x + y - z = 17 \end{cases}$$

d. $$\begin{cases} 2x - 3z = 4 \\ 2x + y - 5z = -1 \\ 3y - 4z = 2 \end{cases}$$

7d. The solution to the system is $\left\{\frac{59}{4}, 12, \frac{17}{2}\right\}$.

$$\begin{bmatrix} 2 & 0 & -3 \\ 2 & 1 & -5 \\ 0 & 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

6a. Sample answer.
$10x - 6y + 2z = 1$
Each variable in the equation is a multiple of the variables in the first equation by a factor 2. However, the constant in the equation is not a multiple of the constant in the first equation.

6b. Sample answer.
$10x - 6y + 2z = -8$
Each variable in the equation is a multiple of the variables in the first equation by a factor of 2. The constant in the equation is also a multiple of the first equation by a factor of 2.

7a. The solution to the system is {5, 1, −6}.

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ -17 \end{bmatrix}$$

7b. The system has no solution.
The variables in the third equation are a multiple of the variables in the second equation by a factor of −2. However, the constant in the third equation is not a multiple of the constant in the second equation by the same factor. Therefore, the system has no solution.

7c. The solution to the system is {3.5, 1.5, −1.5}.

$$\begin{bmatrix} 1 & -4 & 3 \\ 2 & 3 & -5 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 19 \\ 17 \end{bmatrix}$$

Answers

1a. Let x = the number of small fruit baskets sold, let y = the number of medium fruit baskets sold, and let z = the number of large fruit baskets sold.

$$\begin{cases} x + y + z = 80 \\ 15.75x + 25y + 32.5z \\ \quad = 2086.25 \\ 2x = z \end{cases}$$

1b. $$\begin{bmatrix} 1 & 1 & 1 \\ 15.75 & 25 & 32.5 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 80 \\ 2086.25 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 35 \\ 30 \end{bmatrix}$$

1c. The Key Club sold 15 small fruit baskets, 35 medium fruit baskets, and 30 large fruit baskets.

2. Sample answer.
If the system has only two equations in two variables, it may be easier to simply solve using substitution or linear combinations than to enter the coefficient matrix into a calculator, determine its inverse, and then multiply by the constant matrix.

NOTES

TALK the TALK

Show Us Your Stuff

As you learned in a previous lesson, you can model a real-world situation using a system of linear equations. Now you have another tool to use to help you solve the system.

1. **Your school's Key Club decided to sell fruit baskets to raise money for a local charity. The club sold a total of 80 fruit baskets. There were three different types of fruit baskets. Small fruit baskets sold for $15.75 each, medium fruit baskets sold for $25 each, and large fruit baskets sold for $32.50 each. The Key Club took in a total of $2086.25, and they sold twice as many large baskets as small baskets.**

   a. **Formulate a system of three linear equations in three variables to represent this problem situation. Be sure to define your variables.**

   b. **Solve the system of three linear equations using technology with matrices. Write your answer in terms of the problem situation.**

   c. **Describe the solution in terms of the problem situation.**

2. **Is it always easier to solve a system of linear equations using matrices? Explain your reasoning.**