4

Putting the V in Absolute Value

Defining Absolute Value Functions and Transformations

MATERIALS

Graphing technology
Masking tape
Markers

Lesson Overview

Students are already familiar with the general shape of the graphs of absolute value functions, and they have studied transformations of linear functions. In this lesson, students experiment with the absolute value function family. They expand their understanding of transformations to include horizontal translations and dilations. Students interpret functions in the form $f(x) = A(B(x - C)) + D$. They distinguish between the effects of changing values inside the argument of the function (the B- and C-values) and changing values outside the function (the A- and D-values). At the end of the lesson, students summarize the impact of transformations on the domain and range of the absolute value function.

Algebra 2

Attributes of functions and their inverses

(2) The student applies mathematical processes to understand that functions have distinct key attributes and understand the relationship between a function and its inverse. The student is expected to:

(A) graph the functions $f(x) = \sqrt{x}$, $f(x) = \frac{1}{x}$, $f(x) = x^3$, $f(x) = \sqrt[3]{x}$, $f(x) = b^x$, $f(x) = |x|$, and $f(x) = log_b(x)$ where b is 2, 10, and e, and, when applicable, analyze the key attributes such as domain, range, intercepts, symmetries, asymptotic behavior, and maximum and minimum given an interval.

Cubic, cube root, absolute value and rational functions, equations, and inequalities

(6) The student applies mathematical processes to understand that cubic, cube root, absolute value and rational functions, equations, and inequalities can be used to model situations, solve problems, and make predictions. The student is expected to:

(C) analyze the effect on the graphs of $f(x) = |x|$ when $f(x)$ is replaced by $af(x)$, $f(bx)$, $f(x - c)$, and $f(x) + d$ for specific positive and negative real values of a, b, c, and d.

Number and algebraic methods

(7) The student applies mathematical processes to simplify and perform operations on expressions and to solve equations. The student is expected to:

(I) write the domain and range of a function in interval notation, inequalities, and set notation.

ELPS

1.A, 1.D, 1.E, 1.G, 2.C, 2.D, 2.G, 2.H, 2.I, 3.A, 3.B, 3.C, 3.D, 3.F, 4.A, 4.B, 4.C, 4.G, 4.K, 5.E

Essential Ideas

- An absolute value function is a function of the form $f(x) = |x|$.
- A function $g(x)$ of the form $g(x) = f(x) + D$ is a vertical translation of the function $f(x)$. The value $|D|$ describes the number of units the graph of $f(x)$ is translated up or down. If $D > 0$, the graph is translated up; if $D < 0$, the graph is translated down.
- A function $g(x)$ of the form $g(x) = Af(x)$ is a vertical dilation of the function $f(x)$. For $|A| > 1$, the graph is vertically stretched by a factor of A units; for $0 < |A| < 1$, the graph vertically compresses by a factor of A units. For $A < 0$, the graph also reflects across the x-axis.
- A function $g(x)$ of the form $g(x) = f(x - C)$ is a horizontal translation of the function $f(x)$. The value $|C|$ describes the number of units the graph of $f(x)$ is translated right or left. If $C > 0$, the graph is translated to the right; if $C < 0$, the graph is translated to the left.
- A function $g(x)$ of the form $g(x) = f(Bx)$ is a horizontal dilation of the function $f(x)$. For $|B| > 1$, the graph is horizontally compressed by a factor of $\frac{1}{|B|}$. For $0 < |B| < 1$, the graph will be horizontally stretched by a factor of $\frac{1}{|B|}$ units. For $B < 0$, the graph also reflects across the y-axis.
- Transforming a function by changing the A- or D-values affects the output of the function, y. Transforming a function by changing the B- or C-values affects the input of the function, x.

Lesson Structure and Pacing: 3 Days

Day 1

Engage

Getting Started: Distance Is Always Positive

The term *absolute value* is defined, and students recall how to evaluate absolute value expressions prior to exploring absolute value functions in the next activity. They model different absolute value expressions on the x-axis of a classroom coordinate plane. Students also address a common misconception that the expression $-x$ always represents a negative number.

Develop

Activity 4.1: Graphs of Absolute Value Functions

Students model the functions $f(x) = x$, $f(x) = |x|$, $f(x) = -x$ and $f(x) = |-x|$ on a classroom coordinate plane. They record the values for each function in a table, graph them on a coordinate plane, and compare the functions and graphs. Through this activity, students make sense of the basic shape of an absolute value function and its limited range.

Day 2

Activity 4.2: Transformations Outside the Function
Students connect what they know about linear function transformations to translate and dilate absolute value functions. Students explain how the order of transformations affects the line of reflection. For each transformation type, students write the new function in terms of the basic function, graph the function using technology, and then generalize the effect of the *A*-, or *D*-value in the new equation.

Activity 4.3: Transformations Inside the Function
Students investigate horizontal translations, which occur by changing the argument inside absolute value functions. They then connect what they know about vertical dilations to horizontal dilations. For each transformation type, students write the new function in terms of the basic function, graph the function using technology, and then generalize the effect of the *C*-, or *B*-value in the new equation.

Day 3

Activity 4.4: Combining Transformations of Absolute Value Functions
Students distinguish a horizontal translation or dilation, which occurs by changing the *B*- or *C*-values inside the function, from vertical translations and dilations, which occur by changing values outside the function. Given functions of the form $g(x) = A \cdot f(B(x - C)) + D$, students use coordinate notation to describe transformations and then graph each function.

Activity 4.5: Writing Equations in Transformation Form
Students practice writing functions in transformation function form in terms of their specific transformations, then write an equivalent equation for each.

Demonstrate

Talk the Talk: *A, B, C,* and *D*
Students interpret translations of the function $f(x) = A|Bx - C| + D$ by making sense of the values *A, B, C,* and *D* expressed in general terms. They relate the values of the parameters to their graphs. Students also address true/false statements about the effects of transformations on domain and range.

Getting Started: Distance Is Always Positive

Facilitation Notes

In this activity, *absolute value* is defined, and students recall how to evaluate absolute value expressions prior to exploring absolute value functions in the next activity. They model different absolute value expressions on the x-axis of a classroom coordinate plane. Students also address a common misconception that the expression $-x$ always represents a negative number.

Prior to class, use masking tape and a marker to create a coordinate plane on the floor of the classroom. Extend each axis from −8 to 8 and space the grid lines far enough apart so that students can stand at any point.

Complete Questions 1 through 3 as a class. Be sure that students use the coordinate plane to model the expression, not just the final result. For example, for $|-2|$, students should start at −2 and then reflect across 0 to get a result of +2.

Differentiation strategies

- To assist all students, have one student also represent 0 throughout the activity so that the students' distance from zero is more explicit.
- To extend the activity, have students represent these equations on the number line: $|x| = 3$, $|x| = 5$, $|x| = -2$, $|x - 2| = 5$, and $|x + 2| = 4$.

Misconceptions

- Students may incorrectly assume that $-a$ always represents a negative value. To clarify this error in thinking, remind students to read the expression $-x$ as *the opposite of x*.
- Students may have memorized that the answer is always positive in absolute value problems, but it is important that they understand that absolute value is a measure of distance from zero as they deal with the absolute value function in later activities.

As students work, look for

Sign errors when simplifying the expressions.

Questions to ask

- Why did every student go to the positive numbers on the number line?
- Why is the absolute value of an expression always positive or zero?
- How should the equation $|a| = -a$ be read?
- Explain what $|a| = -a$ means in your own words.
- Do you always take the opposite of a number when taking its absolute value? Explain.

Summary

The absolute value of an any numeric expression is a positive number or zero, representing the number of units that value is from zero on the number line.

Activity 4.1
Graphs of Absolute Value Functions

DEVELOP

Facilitation Notes

In this activity, students model the functions $f(x) = x$, $f(x) = |x|$, $f(x) = -x$ and $f(x) = |-x|$ on a classroom coordinate plane. They also record the values for each function in a table, graph them on a coordinate plane, and compare the functions and graphs. Through this activity, students make sense of the basic shape of an absolute value function and its limited range.

Directions for graphing functions on the classroom coordinate plane.

Prior to class:

- Draw both tables from this activity on the board.
- Write $x = -9$, $x = -6$, etc. on separate sheets of paper for each x-value of the table.

During class:

- Select 8 students to represent the x-values from the table. Direct them to hold the paper with their x-value in front of them as they complete this activity.
- Write $f(x) = x$ on the board. Ask the 8 students to plot their coordinate pair for this function on the classroom coordinate plane.
- Have a student ask each student the coordinate pair of their location and record these values in the table drawn on the board while those at their seat complete the table and graph the function.
- Retaining the original eight students, guide students through Questions 1 through 4, repeating this process for $f(x) = |x|$.

Differentiation strategy

Provide students colored pencils so that they can tell the two functions apart when they graph them on their own coordinate plane. Colored pencils will be helpful throughout this lesson as students place multiple graphs on the same coordinate plane.

As students work, look for

A discrete graph rather than a continuous graph. Some graphs may contain only the values in the table rather than all points on the entire function. Emphasize the domain of the function includes all real numbers.

Questions to ask

- What are the domain and range of $f(x) = x$?
- Which students needed to move on the classroom coordinate plane when graphing the absolute value function? Why did they have to move while others did not?
- Which quadrants of the coordinate plane contain positive x-values? Positive y-values?
- Which quadrants contain the absolute value function? Why is that the case?
- What are the domain and range for $f(x) = |x|$?

Complete Questions 5 through 8 as a class, repeating the directions, but selecting different students.

Questions to ask

- Which students needed to move on the classroom coordinate plane when graphing $f(x) = |-x|$? Why did they have to move while others did not?
- How did those students with negative x-values determine their location for the graph of $f(x) = |-x|$?
- How did those students with positive x-values determine their location for the graph of $f(x) = |-x|$?
- What are the domain and range for $f(x) = |-x|$?
- Is an absolute value function also a linear function? Why or why not?
- Are absolute value functions considered increasing functions or decreasing functions? Explain.

Summary

The basic absolute value function is a V-shaped graph. Except for (0, 0), all points on the graph have positive y-coordinates.

Activity 4.2
Transformations Outside the Function

Facilitation Notes

In this activity, students connect what they know about linear function transformations to translate and dilate absolute value functions. Students explain how the order of transformations affects the line of reflection. For each transformation type, students write the new function in terms of the basic function, graph the function using technology, and then generalize the effect of the *A*-, or *D*-value in the new equation.

Differentiation strategy

As an alternative grouping method, use the jigsaw strategy for this activity. The table shown demonstrates how to organize a class of 30 students, with 3 student names per cell.

	Group A Questions 1–4	Group B Questions 5–7
Group 1		
Group 2		
Group 3		
Group 4		
Group 5		

Have each group (column) complete their assigned questions. Then, regroup by group numbers (rows), so that there are groups of 6 students with pairs from Groups A and B. Give each group a time limit to be the teacher to their peers for their set of questions. Then, have all groups read the information prior to Question 8, complete Questions 8 through 15, and discuss as a class.

Have students work with a partner or in a group to complete Questions 1 through 4. Share responses as a class.

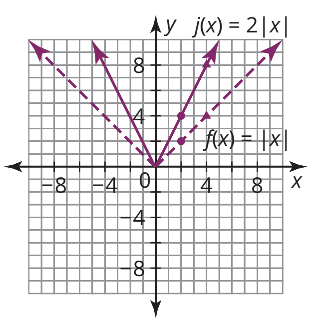
Differentiation strategy

To scaffold support with transferring the graph from their technology to their own coordinate plane, suggest they access the table of values on their technology for help.

Questions to ask

- What is the shape of the absolute value function?
- Where is the absolute value function on your calculator?
- What is a vertical translation?
- How does a vertical translation affect each point on the graph of the original function?
- Does a vertical translation change the shape of a function?
- How do you know whether the vertical translation is a shift up or down?
- How are vertical translations apparent in the algebraic representation of a function?
- Write a function that will translate the absolute value function up 4 units.
- How are vertical translations apparent in the graphic representation of a function?
- Considering a vertical translation of D, does each point (x, y) on the graph of $g(x)$ become $(x, y - D)$ or $(x, y + D)$?

Have students work with a partner or in a group to complete Questions 5 through 7. Share responses as a class.



Misconception

Students may observe the graph of $j(x) = 2|x|$ and identify the change in the graph as a horizontal compression rather than a vertical stretch. Clarify this misconception now using the example provided.

For example, given $j(x) = 2|x|$. The y-value is multiplied by 2.
Point (2, 2) moves to point (2, 4), and point (4, 4) moves to point (4, 8).
It is not the case that point (4, 4) moves to the point (2, 4).

Questions to ask

- What is a vertical dilation?
- How does a vertical dilation affect each point on the graph of the original function?
- Does a vertical dilation change the shape of a function?
- How do you know whether the vertical dilation is a vertical stretch or a vertical compression?
- What dilations cause a vertical stretching of the function?
- What dilations cause a vertical compression of the function?
- How are vertical dilations apparent in the algebraic representation of a function?
- How are vertical dilations apparent in the graphic representation of a function?
- Considering a vertical dilation of factor A, does each point (x, y) on the graph $g(x)$ become (x, Ay) or (Ax, y)?

Have students work with a partner or in a group to complete Questions 8 and 9. Share responses as a class.

Questions to ask

- What is a line of reflection? A line of symmetry?
- What is the difference between a line of reflection and a line of symmetry?
- Are lines of reflection always the *x*- or *y*-axis?
- How are Josh's steps and Vicki's steps different from one another?
- What line of reflection did Josh use? What line of reflection did Vicki use?
- Did Josh and Vicki end up with the same result?
- Does a reflection or a translation change the shape of the function?
- Does a reflection followed by a shift result in the same graph as the same shift followed by a reflection?
- Do you think the order of transformations ever makes a difference in the graph of a function? If so, for what transformations?

Have students work with a partner or in a group to complete Questions 10 through 15. Share responses as a class.

As students work, look for

Errors calculating $b(x)$ by applying the transformations to $a(x)$ rather than $f(x)$.

Misconceptions

- Students may think that all reflections are the same, not realizing the line of reflection makes a difference. Clarify this misconception by comparing $a(x)$ and $-a(x)$, where the *x*-axis is the line of reflection, and $a(x)$ and $b(x)$, where $y = 1$ is the line of reflection. Use patty paper with the axes drawn on it to demonstrate how $b(x)$ and $-a(x)$ are created from reflections of $a(x)$.
- When describing the line of reflection as the *x*-axis, students may identify it as $x = 0$ rather than $y = 0$.

Questions to ask

- Which transformations must be performed on $f(x)$ to create $a(x)$?
- What is the point (2, 2) from $f(x)$ mapped onto in $a(x)$?
- Would the graph of $a(x)$ look different if the vertical translation occurred before the vertical dilation? If so, how?
- What is the point (2, 2) from $f(x)$ mapped onto in $b(x)$?
- Is the graph of $a(x)$ reflected across the *x*-axis to create $b(x)$?
- Is the graph of $a(x)$ reflected across the line $y = 1$ to create $b(x)$?
- What is the line of reflection to create $-a(x)$ from $a(x)$?

Summary

Absolute value functions can be transformed similar to linear functions. Changing the D-value vertically translates the graph of the function. Changing the A-value vertically dilates and/or reflects the graph of the function.

Activity 4.3

Transformations Inside the Function

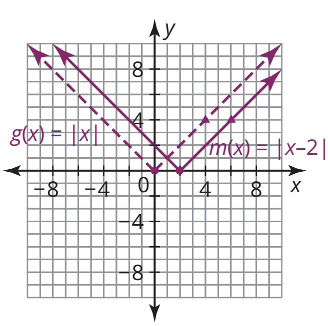
Facilitation Notes

Students investigate horizontal translations, which occur by changing the argument inside absolute value functions. They then connect what they know about vertical dilations to horizontal dilations. For each transformation type, students write the new function in terms of the basic function, graph the function using technology, and then generalize the effect of the C- or B-value in the new equation.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

Misconception

Students may overgeneralize what they know about vertical translations to horizontal translations. They may think that the graph of $m(x) = |x - 2|$ is a horizontal shift from 0 to -2 and that the graph of $n(x) = |x + 2|$ is a horizontal shift of the function from 0 to $+2$. Clarify this misconception by helping students reason through plotting points as explained in the example shown. Follow-up by discussing why the C-value is being subtracted in the function $t(x) = f(x - C)$.



Given $m(x) = |x - 2|$. The expression $x - 2$ means that 2 is subtracted from each x-value before the absolute value function is applied.

Therefore, every value of x must be increased by 2 to get back to the original function.

So, if $x = 2$, the expression inside the absolute value symbol is only 0.

The point (2, 0) in the transformed function corresponds to point (0, 0) in the original function.

The point (0, 0) moves onto point (2, 0). The point (4, 4) moves onto point (6, 4).

Questions to ask

- What is a horizontal translation?
- How does a horizontal translation affect each point on the graph of the original function?
- Does a horizontal translation change the shape of a function?
- What translation causes a shift to the left? To the right?
- How are horizontal translations apparent in the algebraic representation of a function?
- How are horizontal translations apparent in the graphic representation of a function?
- Considering a horizontal translation of C, does each point (x, y) on the graph become $(x + C, y)$ or $(x - C, y)$?

Differentiation strategy

To extend the activity, have students create an informational classroom poster for each transformation of an absolute value function.

Have students work with a partner or in a group to complete Questions 4 through 6. Share responses as a class.

Misconception

Students may observe the graph of $j(x) = |2x|$ and identify the change in the graph as a vertical stretch rather than a horizontal compression. Clarify this misconception now using the example provided. For example, given $j(x) = |2x|$. The x-value is multiplied by $\frac{1}{2}$. Point (2, 2) moves to point (1, 2), and point (4, 4) moves to point (2, 4). It is not the case that point (4, 4) moves to the point (4, 8).

Questions to ask

- What is a horizontal dilation?
- How does a horizontal dilation affect each point on the graph of the original function?
- Does a horizontal dilation change the shape of a function?
- How do you know whether the horizontal dilation is a horizontal stretch or a horizontal compression?
- What dilations cause a horizontal stretching of the function?
- What dilations cause a horizontal compression of the function?
- How are horizontal dilations apparent in the algebraic representation of a function?
- How are horizontal dilations apparent in the graphic representation of a function?
- Considering a horizontal dilation of factor B, does each point (x, y) on the graph $g(x)$ become $(x, \frac{1}{B}y)$ or $(\frac{1}{B}x, y)$?

Summary

Like linear and quadratic functions, absolute value functions can be transformed. Changing the *C*-value horizontally translates the graph of the function. Changing the *B*-value horizontally dilates and/or reflects the graph of the function.

Activity 4.4
Combining Transformations of Absolute Value Functions

Facilitation Notes

In this activity, students distinguish a horizontal translation or dilation, which occurs by changing the *B*- or *C*-value inside the function, from vertical translations and dilations, which occur by changing values outside the function. Given functions of the form $g(x) = A \cdot f(B(x - C)) + D$, students use coordinate notation to describe all three transformations and then graph each function.

Ask a student to read the introduction aloud. Discuss as a class. Have students work with a partner or in a group to complete Question 1. Share responses as a class.

Ask a student to read the paragraph following Question 1 and discuss as a class.

Questions to ask

- How does the ordered pair $(x, |x|)$ describe any point on the basic absolute value function?
- How does the coordinate notation of the transformed absolute value function relate to the coordinate notation you wrote in Question 1?

Have students work with a partner or in a group to complete Questions 2 and 3. Share responses as a class.

As students work, look for

Sign errors associated with transformations that affect the *x*-coordinate (input) of a function and the *y*-coordinate (output) of the function.

Questions to ask

- How does an *A*-value equal to -1 affect the graph of the function?
- How does a *C*-value equal to -1 affect the graph of the function?
- How does a *D*-value equal to -1 affect the graph of the function?
- Which function(s) include a vertical dilation? How can you tell?
- Which function(s) include a vertical translation? How can you tell?
- Which function(s) include a horizontal translation? How can you tell?

- Which function(s) include a reflection? How can you tell?
- What is a general equation for a function that has no vertical translation?
- What is a general equation for a function that has no horizontal translation?
- What is a general equation for a function that has no vertical dilation?

Differentiation strategy

To extend the activity, play *Guess my Function* with the class. Begin by defining a function as $f(x) = |x|$, then write the general equation $g(x) = A \cdot f(B(x - C)) + D$ on the board. Tell students you are thinking of a function that has a vertical translation of -4. Ask them to write this function as an equation. Then tell them this function also is vertically stretched by a factor of 7, and ask that they rewrite their equation to include this transformation. Then tell them the function was also reflected across the x-axis. Ask them to compare their final equations with their classmates' equations. It should be $g(x) = -7 \cdot f(x) - 4$. Play a few rounds of this game for additional practice.

Summary

Given a function of the form $f(x) = A \cdot f(B(x - C)) + D$, a horizontal translation occurs by changing the C-value inside the function argument, while vertical dilations and translations occur by changing the values of A and D, respectively, outside the function. Changing the B-Value results in a horizontal dilation.

Activity 4.5
Writing Equations in Transformation Form

Facilitation Notes

In this activity, students practice writing functions in transformation function form in terms of their specific transformations, then write an equivalent equation for each.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

Questions to ask

- Is a reflection across the x-axis shown in the argument of the function or outside of the function? Why?
- Is a horizontal translation shown in the argument of the function or outside of the function? Why?
- How can you tell whether a vertical dilation is a compression or stretch?
- Why is a vertical compression expressed as a proper fraction?

Summary

Vertical translations, horizontal translations, vertical dilations, and horizontal dilations of functions can be combined to transform absolute value functions.

DEMONSTRATE

Talk the Talk: *A, B, C,* and *D*

Facilitation Notes

In this activity, students interpret translations of the function $f(x) = A \cdot f(B(x - C)) + D$ by making sense of the values *A, B, C, and D* expressed in general terms. They relate the values of the parameters to their graphs. Students also address true/false statements about the effects of transformations on domain and range.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

As students work, look for

- Difficulty dealing with the values of *A, B, C and D* expressed in general terms. If that is the case, suggest students rephrase statements in their own words, such as stating "*D* is positive" or "*A* is a proper fraction."
- Sign errors related to horizontal translations.

Questions to ask

- How does knowing whether *A* is less than 0, between 0 and 1, or greater than 1 help you describe the vertical dilation of a function?
- How does knowing whether *C* is positive or negative help you describe the horizontal translation of a function?
- How does knowing whether *D* is positive or negative help you describe the vertical translation of a function?
- Which values are associated with a vertical translation? Horizontal translation? A stretch? A compression?

Differentiation strategy

To extend the activity, ask students to create a graphical representation to support each of the true statements or rewritten false statements in Question 3.

Summary

Knowing the sign of the values of *A, B, C,* and *D* in $y = A \cdot f(B(x - C)) + D$ helps to inform the type of transformation to be performed on the function.

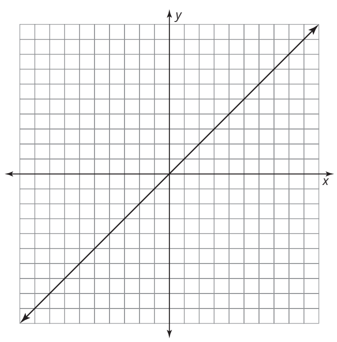
4

Putting the V in Absolute Value

Defining Absolute Value Functions and Transformations

Warm Up

The graph of $f(x) = x$ is shown. Graph each transformation.



1. $g(x) = f(x) + 5$
2. $h(x) = 2 \cdot f(x) - 3$
3. $j(x) = \frac{1}{2} \cdot f(x) - 1$

Learning Goals

- Experiment with transformations of absolute value functions using technology.
- Graph absolute value functions and transformations of absolute value functions.
- Determine the effect of replacing the basic absolute value function $f(x) = |x|$ with $f(x) + D$, $Af(x)$, $f(Bx)$, and $f(x - C)$ for different values of A, B, C, and D.
- Distinguish between function transformations that occur outside the function and inside the argument of the function.

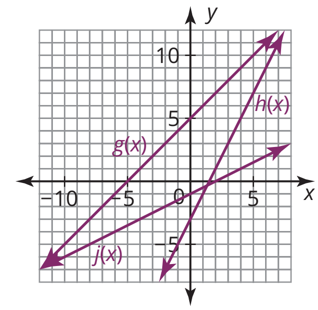
Key Terms

- absolute value
- reflection
- line of reflection
- argument of a function

You know how to transform linear functions. How can you define absolute value functions and show transformations of this function type?

Warm Up Answers

1. See graph.
2. See graph.
3. See graph.



ELL Tip

Review the term *transformation* and the types of transformations students have encountered.

Answers

1a. 2
1b. 2
1c. 1
1d. 2
1e. 6
1f. 0
1g. 4
1h. 2

2. Answers will vary.

   Distance is always positive or zero, so the absolute value of an expression is always positive or zero.

3. Answers will vary.

   If a is a negative number, then $-a = |a|$, e.g. $-(-2) = |-2|$.

GETTING STARTED

Distance Is Always Positive

Absolute value is indicated with vertical bars: $|-4|$ is read as "the absolute value of -4."

The **absolute value** of a number is its distance from zero on the number line.

1. Follow your teacher's instructions to model each absolute value expression on the *x*-axis of a classroom coordinate plane. Rewrite each expression without the absolute value symbol.

a. $|-2|$ **b.** $|2|$

c. $|1 - 2|$ **d.** $|-3 - (-5)|$

e. $|-2 \cdot 3|$ **f.** $|0 \cdot 4|$

g. $\left|\frac{12}{-3}\right|$ **h.** $|8 \div (-4)|$

2. Write your observations about the absolute value expressions you and your classmates modeled on the number line.

3. Provide counterexamples to show why Sonja's statement is incorrect.

Sonja
Absolute values are always positive. So, $|a| = -a$ is not possible.

2 • TOPIC 1: Functions Derived from Linear Relationships

ELL Tip

Provide an example of a *counterexample*. Display the statement, "All quadrilaterals are parallelograms." Discuss why a kite is a *counterexample* to the statement. Then have students explain that a *counterexample* is *an example that disproves an idea, theory, or proposition*. The prefix counter means opposite or against.

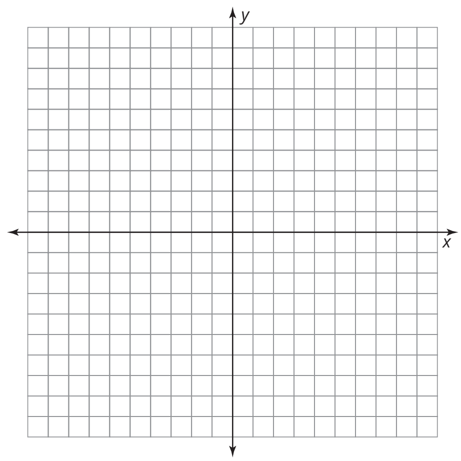
ACTIVITY
4.1

Graphs of Absolute Value Functions

Follow your teacher's instructions to model the function $f(x) = x$ on the classroom coordinate plane with your classmates.

1. Record the coordinates of the plotted points for $f(x) = x$ in the table.

x	y	
	$f(x) = x$	$f(x) = \|x\|$
−9		
−6		
−4		
−1		
0		
3		
5		
8		



2. Change all the plotted points to model the function $f(x) = |x|$. In the table, record the coordinates of the new points for $f(x) = |x|$.

3. Describe how the points move from the graph of $f(x) = x$ to the graph of $f(x) = |x|$.

4. Graph the function $f(x) = |x|$. Describe the characteristics of the function that you notice.

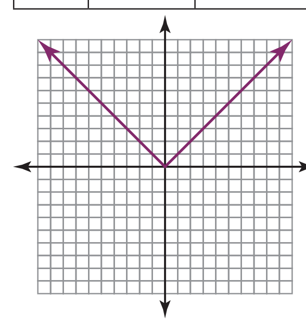
Think about:

What are the domain and range? Is there a maximum value or a minimum value?

Answers

1.

x	y	
	$f(x) = x$	$f(x) = \|x\|$
−9	−9	9
−6	−6	6
−4	−4	4
−1	−1	1
0	0	0
3	3	3
5	5	5
8	8	8



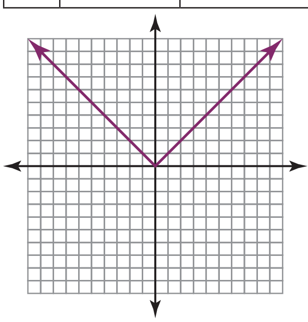
2. See table.
3. The points of the function $f(x) = x$ that have a negative x-coordinate are reflected across the x-axis. The points with positive x-coordinates do not move.
4. Sample answer.

   The graph is *V*-shaped. The vertex is at (0, 0). The domain is all real numbers. The range is $y \geq 0$. The graph is symmetric with respect to the y-axis. The graph has a minimum value.

Answers

5.

x	y	
	$f(x) = -x$	$f(x) = \|-x\|$
−9	9	9
−6	6	6
−4	4	4
−1	1	1
0	0	0
3	−3	3
5	−5	5
8	−8	8

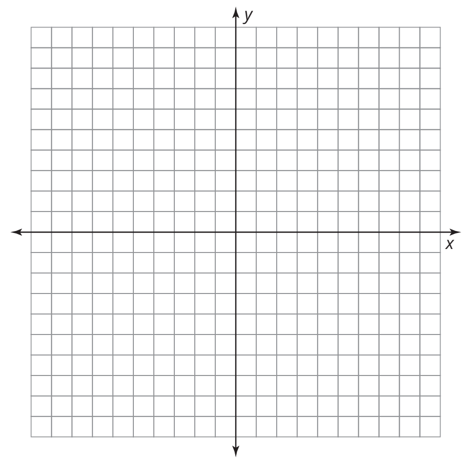


6. See table.
7. The points of the function $f(x) = x$ that have a positive x-coordinate are reflected across the x-axis. The points with negative x-coordinates do not move.
8. The graphs of $f(x) = |x|$ and $f(x) = |-x|$ are the same.

Next, consider the function $f(x) = -x$. Model this function on the classroom coordinate plane with your classmates.

5. Record the coordinates of the plotted points for $f(x) = -x$ in the table.

x	y	
	$f(x) = -x$	$f(x) = \|-x\|$
−9		
−6		
−4		
−1		
0		
3		
5		
8		



6. Change all the plotted points to model the function $f(x) = |-x|$. In the table, record the coordinates of the new points for $f(x) = |-x|$.

7. Describe how the points move from the graph of $f(x) = -x$ to the graph of $f(x) = |-x|$.

8. Graph the function $f(x) = |-x|$. Compare this function with the function $f(x) = |x|$.

Remember:

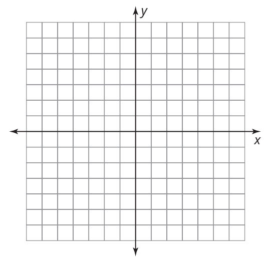
Use a straightedge to be precise when you graph.

ACTIVITY **4.2** Transformations Outside the Function

Consider the three absolute value functions shown.

$g(x) = |x|$ $c(x) = |x| + 3$ $d(x) = |x| - 3$

1. Use technology to graph each function. Then, sketch and label the graph of each function.

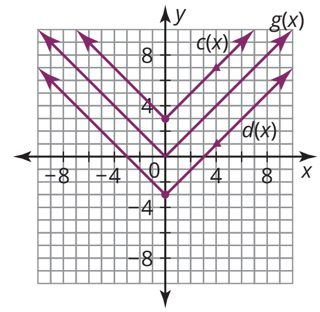


2. Write the functions $c(x)$ and $d(x)$ in terms of the basic function $g(x)$. Then describe the transformations of each function.

3. Describe the similarities and differences between the three graphs. How do these similarities and differences relate to the equations of the functions $g(x)$, $c(x)$, and $d(x)$?

Answers

1.



2. $c(x) = g(x) + 3$; $g(x)$ is translated up 3 units to get $c(x)$.

   $d(x) = g(x) - 3$; $g(x)$ is translated down 3 units to get $d(x)$.

3. Answers will vary.

   The y-intercept of $c(x)$ is 3 more than the y-intercept of $g(x)$. The y-intercept of $d(x)$ is 3 less than the y-intercept of $g(x)$.

   The minimum value of $c(x)$ is 3 more than the minimum value of $g(x)$. The minimum value of $d(x)$ is 3 less than the minimum value of $g(x)$.

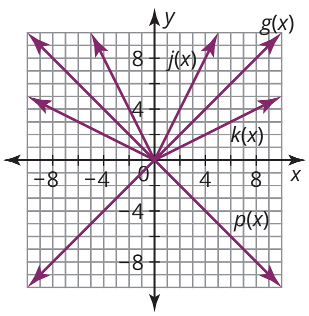
Answers

4a. The function $f(x)$ has the same shape as $g(x)$. The graph of $f(x)$ is a vertical translation of D units up of the function $g(x)$.

4b. The function $f(x)$ has the same shape as $g(x)$. The graph of $f(x)$ is a vertical translation of D units down of the function $g(x)$.

4c. $(x, y + D)$

5.



6. $j(x) = 2g(x)$

$k(x) = \frac{1}{2}g(x)$

$p(x) = -1g(x)$

The graph of $j(x)$ is a vertical stretch of 2 units of the function $g(x)$.

The graph of $k(x)$ is a vertical compression of $\frac{1}{2}$ of the function $g(x)$.

The graph of $p(x)$ is a reflection across the x-axis, or line $y = 0$, of the function $g(x)$.

Recall that a function $t(x)$ of the form $t(x) = f(x) + D$ is a vertical translation of the function $f(x)$. The value $|D|$ describes how many units up or down the graph of the original function is translated.

4. Describe each graph in relation to the basic function $g(x) = |x|$. Then use coordinate notation to represent the vertical translation.

a. $f(x) = g(x) + D$ when $D > 0$

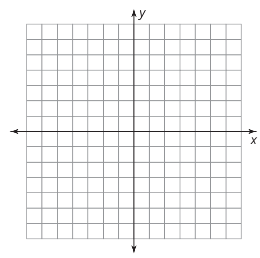
b. $f(x) = g(x) + D$ when $D < 0$

c. Each point (x, y) on the graph of $g(x)$ becomes the point ______________ on $f(x)$.

Consider these absolute value functions.

$g(x) = \lvert x \rvert$	$k(x) = \frac{1}{2}\lvert x \rvert$
$j(x) = 2\lvert x \rvert$	$p(x) = -\lvert x \rvert$

5. Use technology to graph each function. Then, sketch and label the graph of each function.



6. Write the functions $j(x)$, $k(x)$, and $p(x)$ in terms of the basic function $g(x)$. Then describe the transformations of each function.

6 • TOPIC 1: Functions Derived from Linear Relationships

Recall that a function $t(x)$ of the form $t(x) = A \cdot f(x)$ is a vertical dilation of the function $f(x)$. The A-value describes the vertical dilation of the graph of the original function.

NOTES

7. Describe each graph in relation to the basic function $g(x) = |x|$. Then use coordinate notation to represent the vertical dilation.

a. $f(x) = A \cdot g(x)$ when $A > 1$

b. $f(x) = A \cdot g(x)$ when $A < 0$

c. $f(x) = A \cdot g(x)$ when $0 < A < 1$

d. Each point (x, y) on the graph of $g(x)$ becomes the point ____________ on $f(x)$.

A **reflection** of a graph is the mirror image of the graph about a line of reflection.

A **line of reflection** is the line that the graph is reflected across. A horizontal line of reflection affects the y-coordinates.

You know that changing the A-value of a function to its opposite reflects the function across a horizontal line. But the *line of reflection* for the function might be different depending on how you write the transformation and the order the transformations are applied.

Answers

7a. The graph of $f(x)$ is a vertical stretch of $g(x)$.

7b. The graph of $f(x)$ is a reflection across the x-axis and a vertical dilation of $g(x)$.

7c. The graph of $f(x)$ is a vertical compression of $g(x)$.

7d. (x, Ay)

Answers

8. Josh

   Step 1: I graphed the basic function $b(x) = |x|$.

   Step 2: The *A*-value is -1, so I reflected the function across the *x*-axis, or line $y = 0$.

   Step 3: The *D*-value is -3, so I translated the function down 3 units.

   Vicki

   Step 1: I graphed the basic function $b(x) = |x|$.

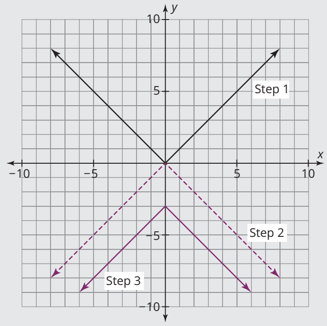
   Step 2: The *D*-value is -3, so I translated the function down 3 units.

   Step 3: The *A*-value is -1, so I reflected the function across the line $y = D$, or $y = -3$.

9. When you interpret and transform a graph according to the *A*-value first, the line of reflection is $y = 0$. When you interpret and transform a graph according to the *D*-value first and then interpret the *A*-value, the line of reflection is $y = D$.

8. Josh and Vicki each sketched a graph of the function $b(x) = -|x| - 3$ using different strategies. Write the step-by-step reasoning used by each student.

Josh

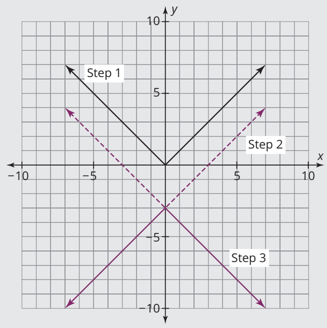


Step 1:

Step 2:

Step 3:

Vicki



Step 1:

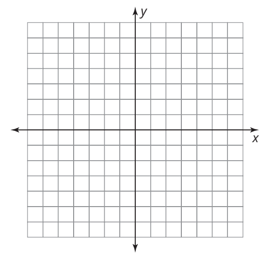
Step 2:

Step 3:

9. Explain how changing the order of the transformations affects the line of reflection.

8 • TOPIC 1: Functions Derived from Linear Relationships

Given the function $f(x) = |x|$. Use the coordinate plane shown to answer Questions 10 through 14.



10. Consider the function $a(x) = 2f(x) + 1$.

a. Use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of $a(x)$.

b. Graph and label $a(x)$ on the coordinate plane shown.

11. Consider the function $b(x) = -2f(x) + 1$.

a. Use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of $b(x)$.

b. Graph and label $b(x)$ on the same coordinate plane shown.

12. Describe the graph of $b(x)$ in terms of $a(x)$.

13. Consider the function $-a(x)$.

a. Use coordinate notation to describe how each point (x, y) on the graph of $a(x)$ becomes a point on the graph of $-a(x)$.

b. Graph and label $-a(x)$ on the coordinate plane shown.

14. Describe the graph of $-a(x)$ in terms of $a(x)$.

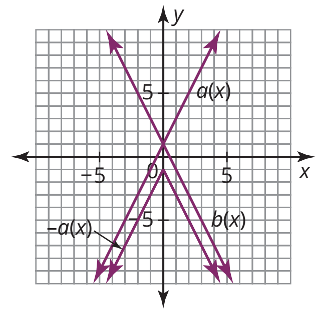
15. How do the A-value and D-value affect the minimum and maximum values of the function?

LESSON 4: Putting the V in Absolute Value • 9

Answers

10a. $(x, 2y + 1)$

10b.



11a. $(x, -2y + 1)$

11b. See graph.

12. The graph of the function $b(x)$ is a reflection of $a(x)$ across the line $y = 1$.

13a. $(x, -2y - 1)$

13b. See graph.

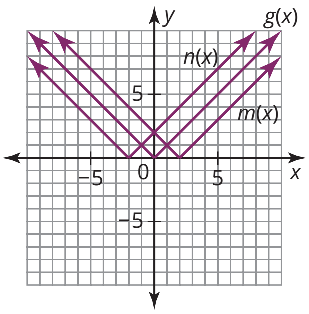
14. The graph of the function $-a(x)$ is a reflection of $a(x)$ across the x-axis, or the line $y = 0$.

15. When $A < 0$, the function has a maximum value at D. When $A > 0$, the function has a minimum value at D.

Answers

1. The graph of $m(x)$ is a translation of two units to the right of $g(x)$.

   The graph of $n(x)$ is a translation of 2 units to the left of $g(x)$.



2. $m(x) = g(x - 2)$

   $n(x) = g(x + 2)$

   Changing the *C*-value to 2 translated the function $g(x)$ horizontally 2 units to the right to create $m(x)$. Changing the *C*-value to −2 translated the function $g(x)$ horizontally 2 units to the left to create $n(x)$.

3. Each point (x, y) of $g(x)$ becomes the point $(x + C, y)$ on the graph of the function that has been horizontally translated.

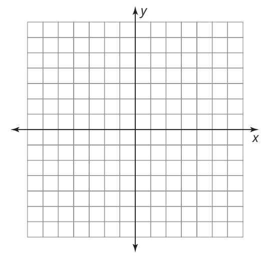
ACTIVITY **4.3**

Transformations Inside the Function

Consider these absolute value functions.

$g(x) = |x|$ $m(x) = |x - 2|$ $n(x) = |x + 2|$

1. Use technology to graph each function. Then, sketch and label the graph of each function. Describe how $m(x)$ and $n(x)$ relate to $g(x)$.



Remember:

The expression $x + C$ is the same as $x - (-C)$.

A function $t(x)$ of the form $t(x) = f(x - C)$ is a horizontal translation of the function $f(x)$. The value $|C|$ describes the number of units the graph of $f(x)$ is translated right or left. If $C > 0$, the graph is translated to the right. If $C < 0$, the graph is translated to the left.

2. Write the functions $m(x)$ and $n(x)$ in terms of the basic function $g(x)$. Describe how changing the *C*-value in the functions $m(x)$ and $n(x)$ horizontally translated the function $g(x)$.

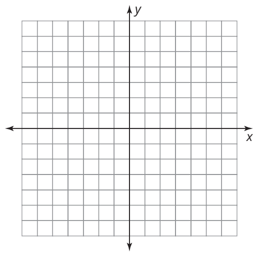
3. Use coordinate notation to show how each point (x, y) on the graph of $g(x)$ becomes a point on a graph that has been horizontally translated.

Consider these absolute value functions.

$g(x) = |x|$ $\qquad$ $k(x) = |\frac{1}{2}x|$

$j(x) = |2x|$ $\qquad$ $p(x) = |-x|$

4. Use technology to graph each function. Then, sketch and label the graph of each function.



5. Write the functions $j(x)$, $k(x)$, and $p(x)$ in terms of the basic function $g(x)$. Then describe the transformations of each function.

Think about:

How does changing the *B*-value compare to changing the *A*-value?

Recall that a function $t(x)$ of the form $t(x) = f(B \cdot x)$ is a horizontal dilation of the function $f(x)$. The *B*-value describes the horizontal dilation of the graph of the original function.

6. Describe each graph in relation to the basic function $g(x) = |x|$. Then use coordinate notation to represent the horizontal dilation.

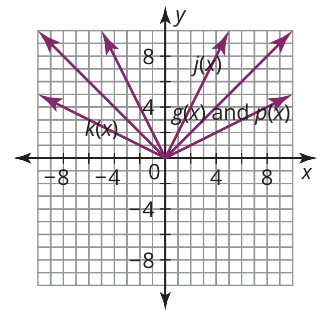
a. $f(x) = g(B \cdot x)$ when $B > 1$

b. $f(x) = g(B \cdot x)$ when $B < 0$

c. $f(x) = g(B \cdot x)$ when $0 < B < 1$

d. Each point (x, y) on the graph of $g(x)$ becomes the point ______________ on $f(x)$.

4.



5. $j(x) = g(2x)$

$k(x) = g(\frac{1}{2}x)$

$p(x) = g(-1x)$

The graph of $j(x)$ is a horizontal compression of $\frac{1}{2}$ units of the function $g(x)$.

The graph of $k(x)$ is a horizontal stretch of 2 units of the function $g(x)$.

The graph of $p(x)$ is a reflection across the y-axis, or line $x = 0$, of the function $g(x)$.

6a. The graph of $f(x)$ is a horizontal compression of $g(x)$.

6b. The graph of $f(x)$ is a reflection across the y-axis and a horizontal dilation of $g(x)$.

6c. The graph of $f(x)$ is a horizontal stretch of $g(x)$.

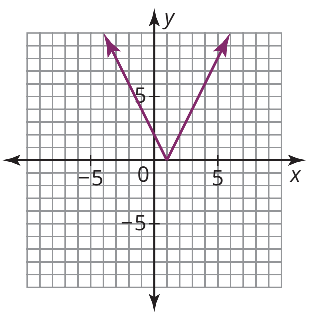
6d. $(\frac{1}{B}x, y)$

Answers

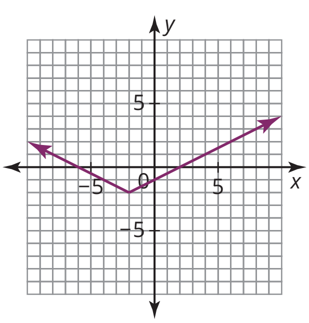
1. $\left(\frac{1}{B}x + C, Ay + D\right)$

2a. $A = 2$ is a transformation that affects the y-values, and $C = 1$ affects the x-values.

The graph of $f(x)$ is vertically stretched by 2 and translated horizontally to the right 1 unit. Each ordered pair (x, y) of $f(x)$ is the new ordered pair $(x + 1, 2y)$ of $m(x)$.



2b. $A = \frac{1}{2}$ and $D = -2$ are both transformations that affect the y-values and $C = -2$ affects the x-values. The graph of $f(x)$ is vertically compressed by $\frac{1}{2}$, translated vertically down 2 units, and translated horizontally to the left 2 units. Each ordered pair (x, y) of $f(x)$ is the new ordered pair $\left(x - 2, \frac{1}{2}y - 2\right)$ of $r(x)$.



ACTIVITY **4.4**

Combining Transformations of Absolute Value Functions

> The **argument of a function** is the expression inside the parentheses.
>
> For $y = f(x - C)$ the expression $x - C$ is the argument of the function.

When a function is transformed by changing the A- or D-values or both, these changes are said to occur "outside the function." These values affect the output to a function, y. When the B- or C-values are changed, this changes the *argument of the function*. A change to the argument of a function is said to happen "inside the function." These values affect the input to a function, x.

outside the function

$$g(x) = A \cdot f(B(x - C)) + D$$

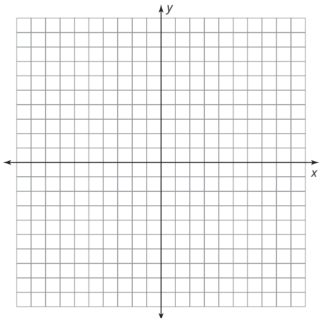
inside the function

1. Use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of $g(x)$.

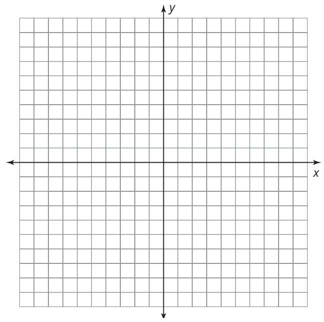
The ordered pair $(x, |x|)$ describes any point on the graph of the basic absolute value function $f(x) = |x|$. For a transformation of the function, any point on the graph of the new function can be written as $\left(\frac{1}{B}x + C, A|(B(x - C))| + D\right)$.

2. Given the basic absolute value function $f(x) = |x|$. Consider each transformation. Describe how the transformations affected $f(x)$. Then use coordinate notation to describe how each point (x, y) on the graph of $f(x)$ becomes a point on the graph of the transformed function. Finally, sketch a graph of each new function.

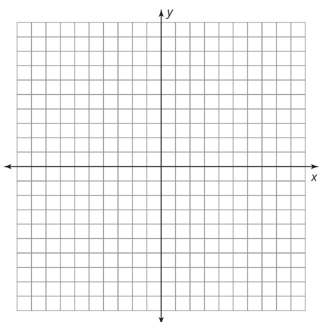
a. $m(x) = 2f(x - 1)$



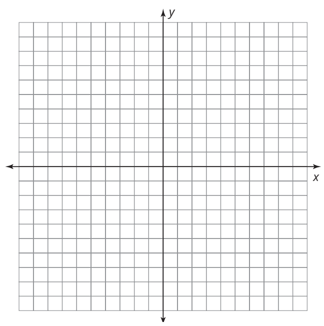
b. $r(x) = \frac{1}{2}f(x + 2) - 2$



c. $w(x) = 2f(x + 3) + 1$



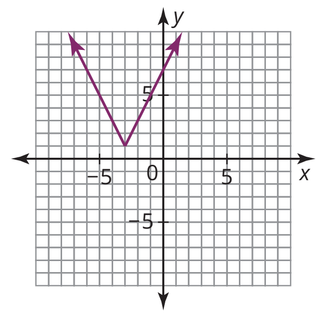
d. $v(x) = -2f(x + 3) + 1$



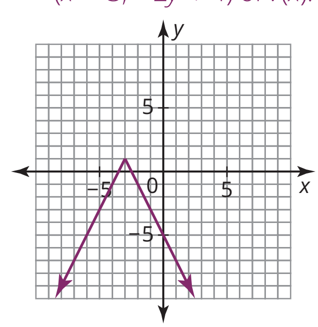
3. Graph $-w(x)$ on the same coordinate plane as $w(x)$ in Question 2 part (c). Describe the similarities and differences between the graph of $v(x)$ and the graph of $-w(x)$.

Answers

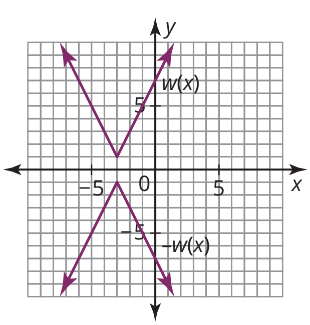
2c. $A = 2$ and $D = 1$ are both transformations that affect the y-values, and $C = -3$ affects the x-values. The graph of $f(x)$ is vertically stretched by 2, translated vertically up 1 unit, and translated horizontally to the left 3 units. Each ordered pair (x, y) of $f(x)$ is the new ordered pair $(x - 3, 2y + 1)$ of $w(x)$.



2d. $A = -2$ and $D = 1$ are both transformations that affect the y-values and $C = -3$ affects the x-values. The graph of $f(x)$ is vertically stretched by 2 and reflected across the line $y = 0$, translated vertically up 1 unit, and translated horizontally to the left 3 units. Each ordered pair (x, y) of $f(x)$ is the new ordered pair $(x - 3, -2y + 1)$ of $v(x)$.



3.



The graph of $v(x)$ is a reflection of the graph of $w(x)$ across the line $y = 1$. The graph of $-w(x)$ is the reflection of the graph of $w(x)$ across the line $y = 0$, or the x-axis.

Answers

1a. $-f(x)$
$y = -|x|$

1b. $f(x + 2) + 3$
$y = |x + 2| + 3$

1c. $-2f(x)$
$y = -2|x|$

1d. $-2f(x) + 3$
$y = -2|x| + 3$

1e. $\frac{1}{2}f(x - 3) - 2$
$y = \frac{1}{2}|x - 3| - 2$

1f. $\frac{1}{4}f(x)$
$y = \frac{1}{4}|x|$

1g. $4f(x)$
$y = 4|x|$

1h. $f(3x)$
$y = |3x|$

ACTIVITY 4.5

Writing Equations in Transformation Form

1. Consider the function, $f(x) = |x|$. Write the function in transformation function form in terms of the transformations described, then write an equivalent equation.

Transformation	Transformation Function Form	Equation
a. Reflection across the *x*-axis		
b. Horizontal translation of 2 units to the left and a vertical translation of 3 units up		
c. Vertical stretch of 2 units and a reflection across the line $y = 0$		
d. Vertical dilation of 2 units and a reflection across the line $y = 3$		
e. Horizontal translation of 3 units to the right, a vertical translation down 2 units, and a vertical dilation of $\frac{1}{2}$		
f. Vertical compression by a factor of 4		
g. Vertical stretch by a factor of 4		
h. Horizontal compression by a factor of 3		

TALK the TALK

A, B, C, and D

The function $f(x) = A|x - C| + D$ is graphed with varying values for A, C, and D.

1. Match the given values of A, C, and D with the graph of the function with corresponding values. Explain your reasoning.

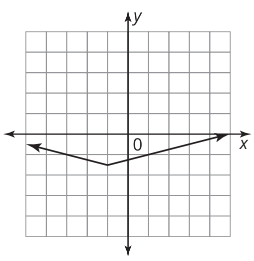
a. $A = 1$, $C = 0$, and $D > 0$

b. $A = 1$, $C = 0$, and $D < 0$

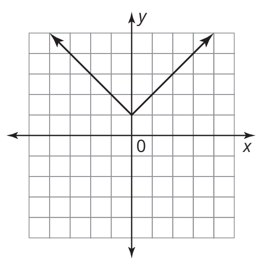
c. $A > 1$, $C > 0$, and $D > 0$

d. $0 < A < 1$, $C < 0$, and $D < 0$

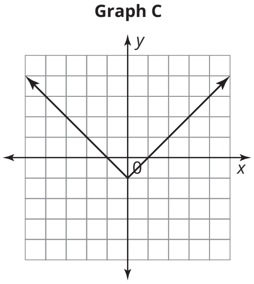
Graph A



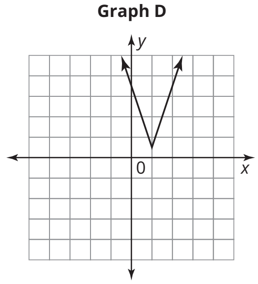
Graph B



Graph C



Graph D



NOTES

Answers

1a. Graph B

1b. Graph C

1c. Graph D

1d. Graph A

Answers

2.

$f(x) = |x| + D$

$D < 0$

Vertical shift down D units.

$D > 0$

Vertical shift up D units.

$f(x) = A|x|$

$A < 0$

Reflection across the x-axis and vertical stretch or compression.

$0 < A < 1$

Vertical compression (dilation by factor A).

$A > 1$

Vertical stretch (dilation by factor A).

$f(x) = |x - C|$

$C < 0$

Horizontal shift left C units.

$C > 0$

Horizontal shift right C units.

$f(x) = |Bx|$

$B < 0$

Reflection across the y-axis and horizontal stretch or compression.

$0 < B < 1$

Horizontal stretch (dilation by a factor of $\frac{1}{B}$).

$B > 1$

Horizontal compression (dilation by a factor of $\frac{1}{B}$).

NOTES

2. Complete the table by describing the graph of each function as a transformation of the basic function $f(x) = |x|$.

Function Form	Equation Information	Description of Transformation
$f(x) = \|x\| + D$	$D < 0$	
	$D > 0$	
$f(x) = A\|x\|$	$A < 0$	
	$0 < A < 1$	
	$A > 1$	
$f(x) = \|x - C\|$	$C < 0$	
	$C > 0$	
$f(x) = \|Bx\|$	$B < 0$	
	$0 < B < 1$	
	$B > 1$	

NOTES

3. **Determine whether each statement is true or false. If the statement is false, rewrite the statement as true.**

   a. **In the transformation function form $g(x) = Af(B(x - C)) + D$, the A-value vertically stretches or compresses $f(x)$, the C-value translates $f(x)$ horizontally, the B-value horizontally stretches or compresses $f(x)$, and the D-value translates the function $f(x)$ vertically.**

   b. **Key characteristics of the basic absolute value function include a domain and range of real numbers.**

   c. **The domain of absolute value functions is not affected by translations or dilations.**

   d. **Vertical translations do not affect the range of absolute value functions.**

   e. **Horizontal translations do not affect the range of absolute value functions.**

   f. **Vertical dilations do not affect the range of absolute value functions.**

Answers

3a. true

3b. false
The range for $y = |x|$ is $y \geq 0$.

3c. true

3d. false
Vertical translations affect the y-values.

3e. true

3f. true