Extending Linear Relationships Summary

KEY TERMS

- Gaussian elimination
- solution of a system of linear inequalities
- linear programming
- matrix (matrices)
- dimensions
- square matrix
- matrix element
- matrix multiplication
- identity matrix
- multiplicative inverse of a matrix
- matrix equation
- coefficient matrix
- variable matrix
- constant matrix
- absolute value
- reflection
- line of reflection
- argument of a function
- linear absolute value equation
- linear absolute value inequality
- equivalent compound inequality

LESSON 1 Gauss in Das Haus

Recall that a system of two linear equations can be solved algebraically by using either the substitution method or the linear combinations method. Systems of two linear equations can have one solution, no solution, or an infinite number of solutions.

A system of two equations involving one linear equation and one quadratic equation can be solved using methods similar to those for solving a system of two linear equations. These systems can have one solution, two solutions, or no solutions. The solution(s) can be verified by graphing the equations on the same coordinate grid and then calculating the point(s) of intersection.

For example, you can solve the following system of two equations in two variables algebraically and then verify the solution graphically.

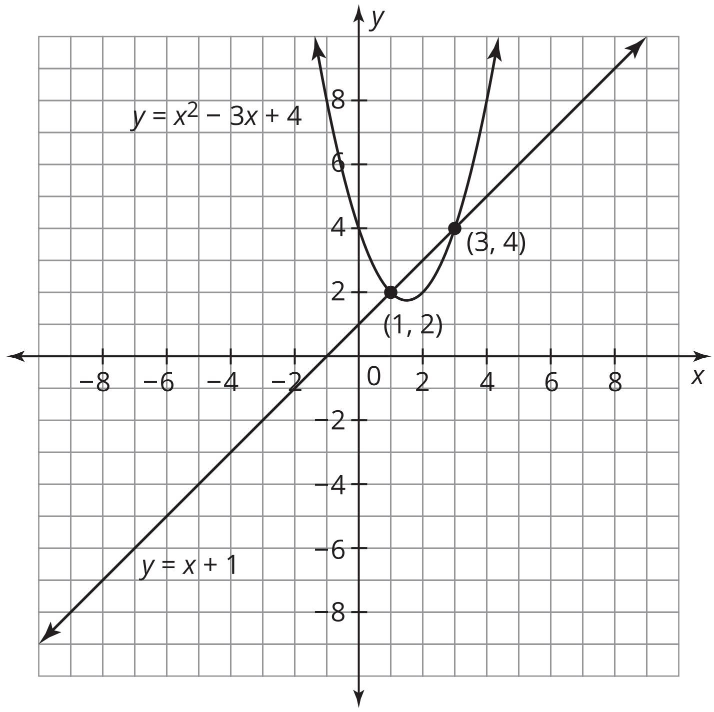
$$\begin{cases} y = x + 1 \\ y = x^2 - 3x + 4 \end{cases}$$

$$x^2 - 3x + 4 = x + 1$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$

$x = 3$ or $x = 1$

Substitute $x = 3$ into the linear equation.
$y = 3 + 1 = 4$
Substitute $x = 1$ into the linear equation.
$y = 1 + 1 = 2$

The solutions to the system are (3, 4) and (1, 2).



To solve a system of three linear equations using substitution, the first step is to solve for one variable in one of the equations. Then substitute this expression for that variable in the other two equations. The two new equations will then have only two unknown variables and can be solved using either substitution or linear combinations.

The goal of **Gaussian elimination** is to use linear combinations to isolate one variable for each equation. When using this method, you can:

- swap the positions of two equations.
- multiply an equation by a nonzero constant.
- add one equation to the multiple of another.

For example, you can solve the system $\begin{cases} x + 5y - 6z = 24 \\ -x - 4y + 5z = -21 \\ 5x - 4y + 2z = 21 \end{cases}$ using Gaussian elimination.

Add the first and second equation and replace the second equation.

$$\begin{array}{r} x + 5y - 6z = 24 \\ -x - 4y + 5z = -21 \\ \hline y - z = 3 \end{array} \quad \rightarrow \quad \begin{cases} x + 5y - 6z = 24 \\ y - z = 3 \\ 5x - 4y + 2z = 21 \end{cases}$$

Multiply the first equation by −5 and add it to the third equation. Replace the third equation.

$$\begin{array}{r} -5x - 25y + 30z = -120 \\ 5x - 4y + 2z = 21 \\ \hline -29y + 32z = -99 \end{array} \quad \rightarrow \quad \begin{cases} x + 5y - 6z = 24 \\ y - z = 3 \\ -29y + 32z = -99 \end{cases}$$

Multiply the second equation by 29 and add it to the third equation. Replace the third equation.

$$\begin{array}{r} 29y - 29z = 87 \\ -29y + 32z = -99 \\ \hline 3z = -12 \end{array} \quad \rightarrow \quad \begin{cases} x + 5y - 6z = 24 \\ y - z = 3 \\ 3z = -12 \end{cases}$$

Multiply the third equation by $\frac{1}{3}$ and add it to the second equation. Replace the second equation.

$$\begin{array}{r} y - z = \ \ 3 \\ z = -4 \\ \hline y = \ -1 \end{array} \rightarrow \begin{cases} x + 5y - 6z = 24 \\ y = -1 \\ 3z = -12 \end{cases}$$

Multiply the second equation by −5 and add it to the first equation. Replace the first equation.

$$\begin{array}{r} x + 5y - 6z = 24 \\ -5y \qquad = \ \ 5 \\ \hline x \qquad - 6z = \ 29 \end{array} \rightarrow \begin{cases} x - 6z = 29 \\ y = -1 \\ 3z = -12 \end{cases}$$

Multiply the third equation by 2 and add it to the first equation. Replace the first equation.

$$\begin{array}{r} x - 6z = \ \ 29 \\ 6z = -24 \\ \hline x \qquad = \ \ \ 5 \end{array} \rightarrow \begin{cases} x = 5 \\ y = -1 \\ 3z = -12 \end{cases}$$

Multiply the third equation by $\frac{1}{3}$. Replace the third equation.

$$\begin{array}{r} \frac{1}{3}(3z = -12) \\ z = -4 \end{array} \rightarrow \begin{cases} x = 5 \\ y = -1 \\ z = -4 \end{cases}$$

The solution to the system is $(x, y, z) = (5, -1, -4)$.

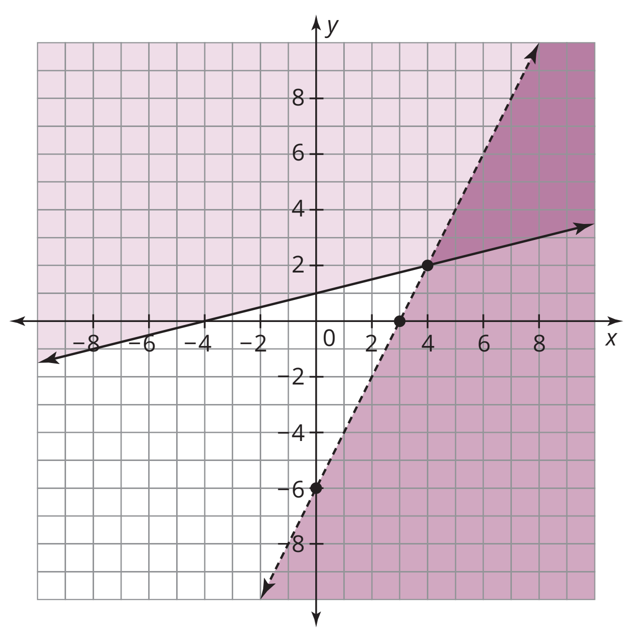
LESSON
2 Make the Best of It

The **solution of a system of inequalities**, is the intersection of the solutions to each inequality. Every point in the intersection satisfies all inequalities in the system.

For example, consider the system $\begin{cases} y < 2x - 6 \\ y \geq \frac{1}{4}x + 1 \end{cases}$.

Graph the system of linear inequalities on one coordinate plane.

The solution to the system of linear inequalities is the overlapping region of the solutions to each inequality in the system. The points (6, 4) and (8, 8) are both solutions to the system of linear inequalities.



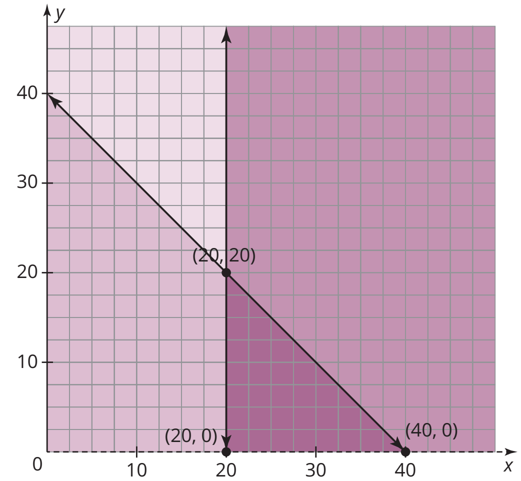
In **linear programming**, the vertices of the solution region of the system of linear inequalities are substituted into an equation to find the maximum or minimum value.

For example, consider the situation in which Anabelle works as a lifeguard and as tutor over the summer. She can work no more than 40 hours each week. Her lifeguarding job requires that she work at least 20 hours each week. If Anabelle earns \$15 for each hour she lifeguards and \$25 for each hour she tutors, how many hours should she work at each job to maximize her earnings?

The situation can be modeled by a system of linear inequalities. Let x represent the number of hours lifeguarding and let y represent the number of hours tutoring.

$$\begin{cases} x + y \leq 40 \\ x \geq 20 \\ x > 0 \\ y > 0 \end{cases}$$

The vertices of the solution region are (20, 0), (20, 20), and (40, 0).



Substitute the coordinates into an equation representing Anabelle's total earnings for both jobs.

$15x + 25y = E$
$15(20) + 25(0) = 300$
$15(20) + 20(20) = 700$
$15(40) + 25(0) = 600$

Anabelle should work 20 hours as a lifeguard and 20 hours as a tutor to maximize her earnings, which would be \$700.

LESSON 3 Systems Redux

The **identity matrix**, I, is a square matrix such that for any matrix A, $AI = A$. The elements of the identity matrix are all zeros except for a diagonal from the upper left to the lower right where the elements are all ones. The **multiplicative inverse of a matrix**, A, labeled as A^{-1}, is a matrix such that when matrix A is multiplied by it the result is the identity matrix. In symbols, $A \cdot A^{-1} = I$. Technology can be used to find the inverse of a square matrix. Not every square matrix has an inverse, and non-square matrices do not have inverses.

For example, you can use matrix multiplication to verify that matrix J^{-1} is the inverse of matrix J.

$$J = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \qquad J^{-1} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} (-2)\left(-\frac{3}{2}\right) + 1(-2) & (-2)\left(\frac{1}{2}\right) + 1(1) \\ (-4)\left(-\frac{3}{2}\right) + 3(-2) & (-4)\left(\frac{1}{2}\right) + 3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 - 2 & -1 + 1 \\ 6 - 6 & -2 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since the product matrix $J \cdot J^{-1}$ yields the identity matrix, J^{-1} is the inverse of matrix J.

A system of linear equations can be written as a **matrix equation**, in the form $A \cdot X = B$, where A is the coefficient matrix, X is the variable matrix, and B is the constant matrix. A **coefficient matrix** is a square matrix that consists of each coefficient of each equation in the system of equations, in order, when they are written in standard form. A **variable matrix** is a matrix in one column that represents all of the variables in the system of equations. A **constant matrix** is a matrix in one column that represents each of the constants in the system of equations.

In a system with n equations, the coefficient matrix will be an $n \times n$ matrix, the variable matrix will be an $n \times 1$ matrix, and the constant matrix will be an $n \times 1$ matrix.

For example, the system of equations $\begin{cases} 2x + 8y + 3z = -21 \\ 7x - 5y + z = 72 \\ -6x - 2y + 3z = 17 \end{cases}$ can be written as the matrix equation shown.

$$\underset{A}{\begin{bmatrix} 2 & 8 & 3 \\ 7 & -5 & 1 \\ -6 & -2 & 3 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} -21 \\ 72 \\ 17 \end{bmatrix}}$$

To solve the matrix equation $A \cdot X = B$, multiply both sides by the inverse of matrix A, or A^{-1}. You can use technology with matrices to calculate the solution.

$$A^{-1} = \begin{bmatrix} \frac{13}{374} & \frac{15}{187} & -\frac{23}{374} \\ \frac{27}{324} & -\frac{12}{187} & -\frac{19}{374} \\ \frac{2}{17} & \frac{2}{17} & \frac{3}{17} \end{bmatrix}$$

$$X = \quad A^{-1} \quad \cdot \quad B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{13}{374} & \frac{15}{187} & -\frac{23}{374} \\ \frac{27}{324} & -\frac{12}{187} & -\frac{19}{374} \\ \frac{2}{17} & \frac{2}{17} & \frac{3}{17} \end{bmatrix} \begin{bmatrix} -21 \\ 72 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix}$$

The solution to the system is (4, −7, 9).

LESSON **4**

Putting the V in Absolute Value

The absolute value of a number is its distance from zero on the number line. Absolute value is indicated with vertical bars: $|-4| = 4$.

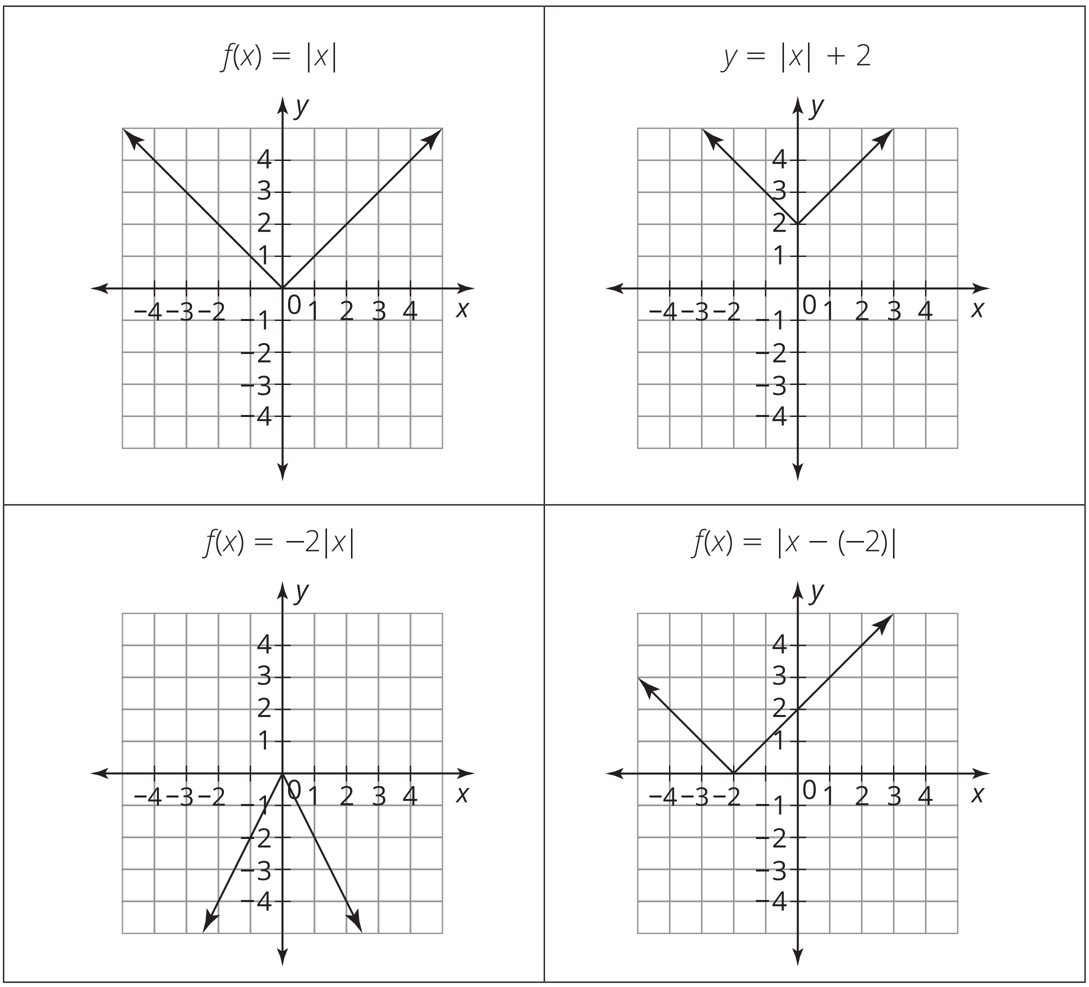
The graph of the absolute value function $f(x) = |x|$ is in a *V* shape. It is the combination of the graphs of $f(x) = x$ and $f(x) = -x$.

For the basic function $f(x) = |x|$, the transformed function $g(x) = f(x) + D$ is a vertical translation of the function $f(x)$. For $D > 0$, the graph vertically shifts up. For $D < 0$, the graph vertically shifts down. The amount of shift is given by $|D|$.

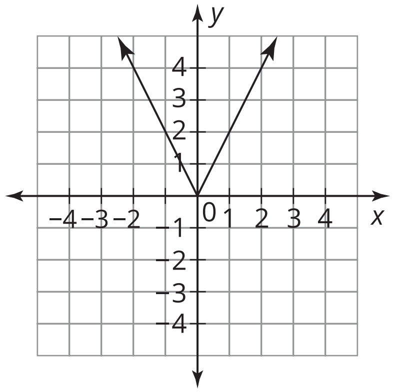
For the basic function $f(x) = |x|$, the transformed function $g(x) = A \cdot f(x)$ is a vertical dilation of the function $f(x)$. For $|A| > 1$, the graph vertically stretches by a factor of A units. For $0 < |A| < 1$, the graph vertically compresses by a factor of A units. For $A < 0$, the graph reflects across a horizontal line. A **reflection** of a graph is the mirror image of the graph about a line of reflection. A **line of reflection** is the line that the graph is reflected across. A horizontal line of reflection affects the *y*-coordinates.

For the basic function $f(x) = |x|$, the transformed function $g(x) = f(x - C)$ is a horizontal translation of the function $f(x)$. For $C > 0$, the graph horizontally shifts right. For $C < 0$, the graph horizontally shifts left. The amount of shift is given by $|C|$.

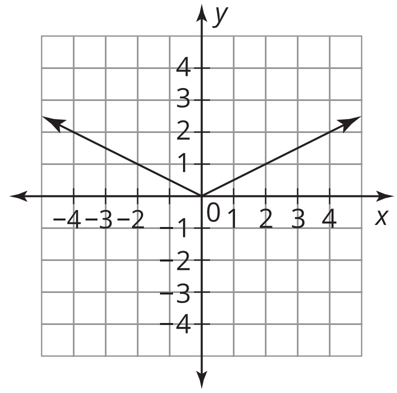
For the basic function $f(x) = |x|$, the transformed function $g(x) = f(Bx)$ is a horizontal dilation of the function $f(x)$. For $|B| > 1$, the graph is horizontally compressed by a factor of $\frac{1}{|B|}$. For $0 < |B| < 1$, the graph will be horizontally stretched by a factor of $\frac{1}{|B|}$ units. For $B < 0$, the graph also reflects across the y-axis.



$f(x) = |2x|$



$f(x) = |\frac{1}{2}x|$



LESSON

5

Play Ball!

To solve a **linear absolute value equation**, write the positive and negative equations that the linear absolute value equation represents. Then solve each equation.

$$|5x - 4| = 21$$

$$+(5x - 4) = 21 \qquad -(5x - 4) = 21$$

$$5x - 4 = 21 \qquad 5x - 4 = -21$$

$$5x - 4 + 4 = 21 + 4 \qquad 5x - 4 + 4 = -21 + 4$$

$$5x = 25 \qquad 5x = -17$$

$$\frac{5x}{5} = \frac{25}{5} \qquad \frac{5x}{5} = \frac{(-17)}{5}$$

$$x = 5 \qquad x = -3\frac{2}{5}$$

If there is a range of solutions that satisfy a problem situation, you can write a **linear absolute value inequality**. To evaluate for a specific value, substitute the value for the variable.

For example, suppose a swimmer who wants to compete on the green team at the City Swim Club should be able to swim the 100-meter freestyle in 54.24 seconds plus or minus 1.43 seconds. Can a swimmer with a time of 53.15 seconds qualify for the green team?

$$|t - 54.24| \leq 1.43$$

$$|53.15 - 54.24| \leq 1.43$$

$$|-1.09| \leq 1.43$$

$$1.09 \leq 1.43$$

The swimmer qualifies because his time is less than 1.43 seconds from the base time.

Absolute value inequalities can take four different forms with the absolute value expression compared to a value, c. To solve an absolute value inequality, you must first write it as an **equivalent compound inequality**. "Less than" inequalities will be conjunctions and "greater than" inequalities will be disjunctions.

Absolute Value Inequality	Equivalent Compound Inequality
$\|ax + b\| < c$	$-c < ax + b < c$
$\|ax + b\| \leq c$	$-c \leq ax + b \leq c$
$\|ax + b\| > c$	$ax + b < -c$ or $ax + b > c$
$\|ax + b\| \geq c$	$ax + b \leq -c$ or $ax + b \geq c$