4

True to Form

Forms of Quadratic Functions

Warm Up

Identify the axis of symmetry of each quadratic function.

1. $f(x) = -4(x - 3)^2 + 2$

2. $g(x) = 2(x + 5)(x - 7)$

3. $h(x) = 4x^2 + 6x + 1$

Learning Goals

- Determine how many points are necessary to create a unique quadratic equation.
- Match a quadratic function with its corresponding graph.
- Identify key characteristics of quadratic functions based on the form of the function.
- Analyze the different forms of quadratic functions.
- Use key characteristics of specific forms of quadratic functions to write equations.
- Derive a quadratic equation given three points using a system of equations.
- Write quadratic functions to represent problem situations.

Key Terms

- standard form of a quadratic function
- factored form of a quadratic function
- vertex form of a quadratic function
- concavity of a parabola

You have explored the key characteristics of different forms of quadratic functions. How can you use these characteristics to write a quadratic equation to model the graph of a parabola, even if you only know certain points on the graph?

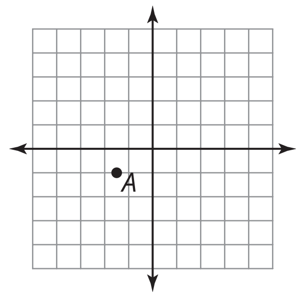
GETTING STARTED

Be My One and Only

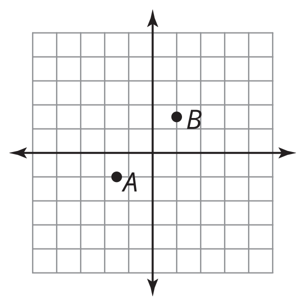
Consider the family of linear functions.

1. Use the given point(s) to sketch possible solutions.

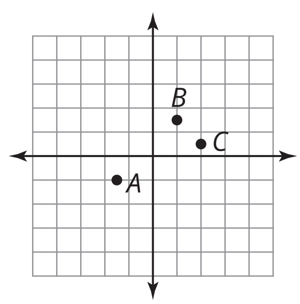
a. How many lines can you draw through point *A*?



b. How many lines can you draw through both points *A* and *B*?



c. How many lines can you draw through all points *A*, *B*, and *C*?

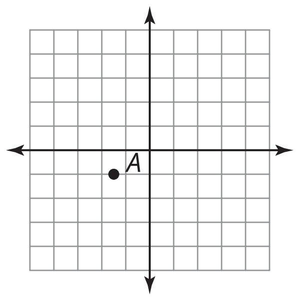


2. What is the minimum number of points you need to draw a unique line?

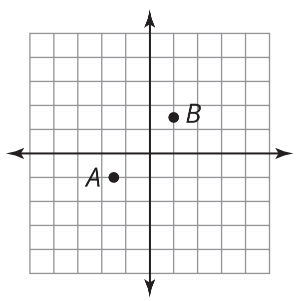
Consider the family of quadratic functions.

3. Use the given point(s) to sketch possible solutions.

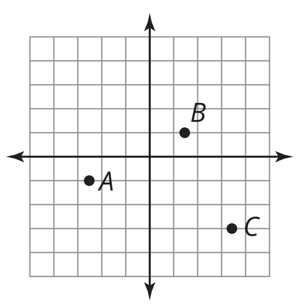
a. How many parabolas can you draw through point *A*?



b. How many parabolas can you draw through both points *A* and *B*?



c. How many parabolas can you draw through all points *A*, *B*, and *C*?



NOTES

ACTIVITY

4.1 Forms of Quadratic Functions

Recall that quadratic functions can be written in different forms.

Standard form: $f(x) = ax^2 + bx + c$, where a does not equal 0

Factored form: $f(x) = a(x - r_1)(x - r_2)$, where a does not equal 0

Vertex form: $f(x) = a(x - h)^2 + k$, where a does not equal 0

The graphs of quadratic functions can be described using key characteristics: x-intercept(s), y-intercept, vertex, axis of symmetry, and concavity.

Concavity of a parabola describes whether a parabola opens up or opens down. A parabola is concave up if it opens upward; a parabola is concave down if it opens downward.

1. The form of a quadratic function reveals different key characteristics. State the characteristics you can determine from each form.

a. Standard form

b. Factored form

C. Vertex form

2. **Christine and Kate were asked to determine the vertex of two different quadratic functions, each written in a different form. Analyze their calculations.**

Christine

$f(x) = 2x^2 + 12x + 10$

The quadratic function is in standard form. So I know the axis of symmetry is $x = \frac{-b}{2a}$.

$$x = \frac{-12}{2(2)}$$
$$= -3$$

Now that I know the axis of symmetry, I can substitute that value into the function to determine the y-coordinate of the vertex.

$$f(-3) = 2(-3)^2 + 12(-3) + 10$$
$$= 2(9) - 36 + 10$$
$$= 18 - 36 + 10$$
$$= -8$$

Therefore, the vertex is $(-3, -8)$.

Kate

$g(x) = \frac{1}{2}(x + 3)(x - 7)$

The form of the function tells me the x-intercepts are -3 and 7. I also know the x-coordinate of the vertex is directly in the middle of the x-intercepts. So, all I have to do is calculate their average.

$$x = \frac{-3 + 7}{2}$$
$$= \frac{4}{2} = 2$$

Now that I know the x-coordinate of the vertex, I can substitute that value into the function to determine the y-coordinate.

$$g(2) = \frac{1}{2}(2 + 3)(2 - 7)$$
$$= \frac{1}{2}(5)(-5)$$
$$= -12.5$$

Therefore, the vertex is $(2, -12.5)$.

a. **How are these methods similar? How are they different?**

b. **What must Kate do to use Christine's method?**

c. **What must Christine do to use Kate's method?**

Do not use technology for this activity. Instead, use the information you can determine by analyzing each equation and each graph to determine a match.

3. Cut out each quadratic equation and graph located at the end of the lesson.

a. Tape each quadratic equation to its corresponding graph.

b. Explain the method(s) you used to match each equation with its graph.

4. Analyze the three tables located at the end of the lesson. Tape each function and its corresponding graph from Question 3 in the "Graphs and Their Functions" section of the appropriate table. Then, explain how you can determine each key characteristic based on the form of the given function.

ACTIVITY
4.2

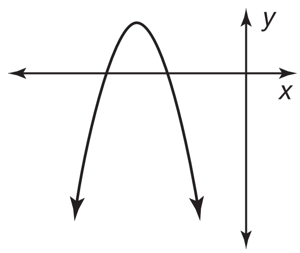
Modeling a Quadratic Graph

1. Analyze each graph. Then, circle the function(s) which could model the graph. Describe the reasoning you used to either eliminate or choose each function.

Think about:

What information is given by each function and the relative position of its graph?

a.



$f_1(x) = -2(x + 1)(x + 4)$

$f_2(x) = -\frac{1}{3}x^2 - 3x - 6$

$f_3(x) = 2(x + 1)(x + 4)$

$f_4(x) = 2x^2 - 8.9$

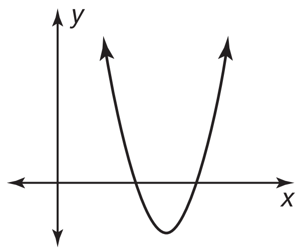
$f_5(x) = 2(x - 1)(x - 4)$

$f_6(x) = -(x - 6)^2 + 3$

$f_7(x) = -3(x + 2)(x - 3)$

$f_8(x) = -(x + 6)^2 + 3$

b.



$f_1(x) = 2(x - 75)^2 - 92$

$f_2(x) = (x - 8)(x + 2)$

$f_3(x) = 8x^2 - 88x + 240$

$f_4(x) = -3(x - 1)(x - 5)$

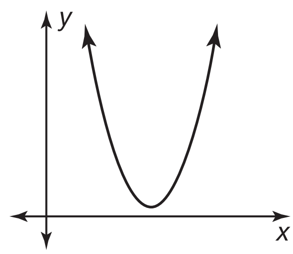
$f_5(x) = -2(x - 75)^2 - 92$

$f_6(x) = x^2 + 6x - 2$

$f_7(x) = 2(x + 4)^2 - 2$

$f_8(x) = (x + 1)(x + 3)$

c.



$f_1(x) = 3(x + 1)(x - 5)$ $f_2(x) = 2(x + 6)^2 - 5$ $f_3(x) = 4x^2 - 400x + 10{,}010$

$f_4(x) = 3(x + 1)(x + 5)$ $f_5(x) = 2(x - 6)^2 + 5$ $f_6(x) = x^2 + 2x - 5$

2. **Consider the two functions shown from Question 1. Identify the form of the function given, and then write the function in the other two forms, if possible. If it is not possible, explain why.**

   a. $f_1(x) = -2(x + 1)(x + 4)$

   b. $f_5(x) = 2(x - 6)^2 + 5$

ACTIVITY

4.3 Writing a Unique Quadratic Function

You have used properties of linear functions to write linear equations. In this activity, you will use properties of quadratic functions to write quadratic equations in various forms.

1. George and Pat each wrote a quadratic equation with a vertex of (4, 8). Analyze each student's work. Describe the similarities and differences in their equations and determine who is correct.

George

$y = a(x - h)^2 + k$

$y = a(x - 4)^2 + 8$

$y = -\frac{1}{2}(x - 4)^2 + 8$

Pat

$y = a(x - h)^2 + k$

$y = a(x - 4)^2 + 8$

$y = (x - 4)^2 + 8$

You can write a unique quadratic function given a vertex and a point on the parabola.

Worked Example

Write the quadratic function given the vertex (5, 2) and the point (4, 9).

Substitute the given values into the vertex form of the function.

$f(x) = a(x - h)^2 + k$

$9 = a(4 - 5)^2 + 2$

Then solve for a.

$9 = a(-1)^2 + 2$

$9 = 1a + 2$

$7 = 1a$

$7 = a$

Finally, substitute the a-value into the function.

$f(x) = 7(x - 5)^2 + 2$

You can write a unique quadratic function given the roots and a point on the parabola.

Worked Example

Write a quadratic function given the roots (–2, 0) and (4, 0), and the point (1, 6).

Substitute the given values into the factored form of the function.	$f(x) = a(x - r_1)(x - r_2)$ $6 = a(1 - (-2))(1 - 4)$
Then solve for *a*.	$6 = a(1 + 2)(1 - 4)$ $6 = a(3)(-3)$ $6 = -9a$ $-\frac{2}{3} = a$
Finally, substitute the *a*-value into the function.	$f(x) = -\frac{2}{3}(x + 2)(x - 4)$

2. Determine which form of a quadratic function would be most efficient to write the function using the given information. Write *standard form, factored form, vertex form,* or *none* in the space provided.

a. minimum point (6, –75)
***y*-intercept (0, 15)** ____________________

b. points (2, 0), (8, 0), and (4, 6) ____________________

c. points (100, 75), (450, 75), and (150, 95) ____________________

d. points (3, 3), (4, 3), and (5, 3) ____________________

e. *x*-intercepts (7.9, 0) and (–7.9, 0)
point (–4, –4) ____________________

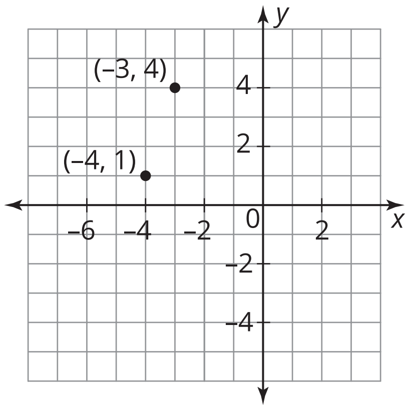
f. roots (3, 0) and (12, 0)
point (10, 2) ____________________

g. Max hits a baseball off a tee that is 3 feet high. The ball reaches a maximum height of 20 feet when it is 15 feet from the tee. ____________________

3. **Write a quadratic function that includes the given points. If it is not possible to write a function, state why not.**

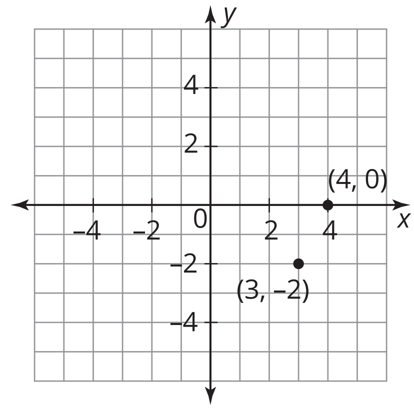
a. **Given: vertex (−3, 4); point (−4, 1)**

$f(x) =$ ______________________



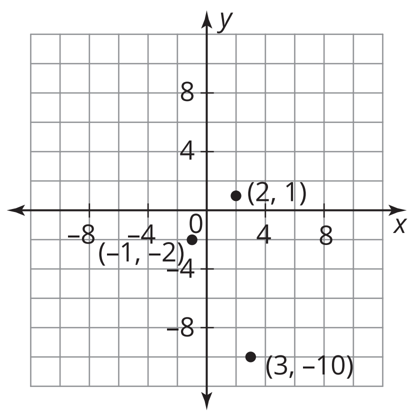
b. **Given: vertex (3, −2); one of two *x*-intercepts (4, 0)**

$f(x) =$ ______________________



c. **Given: points (2, 1), (−1, −2), (3, −10)**

$f(x) =$ ______________________



4. **Wilhemina says that she can write a unique quadratic function given only two points. Is she correct? Explain your reasoning.**

ACTIVITY 4.4

Using Algebra to Write a Quadratic Equation

In the previous activity, there were times when you could not determine a quadratic equation using known strategies. You know how to use technology to determine the equation, but what if you don't have technology?

You know that you need a minimum of 3 non-linear points to create a unique parabola. To create an equation to represent the parabola, you can use a system of equations.

Worked Example

Consider the three points given in Question 3 part (c) in the previous activity: A (2, 1), B (−1, −2), and C (3, −10).

To write an equation in standard form to represent a parabola that passes through three given points, begin by substituting the x- and y-values of each point into $y = ax^2 + bx + c$.

Point A: $1 = a(2)^2 + b(2) + c$
$1 = 4a + 2b + c$

Equation A: $1 = 4a + 2b + c$

Point B: $-2 = a(-1)^2 + b(-1) + c$
$-2 = a - b + c$

Equation B: $-2 = a - b + c$

Point C: $-10 = a(3)^2 + b(3) + c$
$-10 = 9a + 3b + c$

Equation C: $-10 = 9a + 3b + c$

Now, use linear combinations and substitution to solve for a, b, and c.

STEP 1: Subtract Equation B from A:

$$\begin{array}{r} 1 = 4a + 2b + c \\ -(-2 = a - b + c) \\ \hline 3 = 3a + 3b \end{array}$$

STEP 2: Subtract Equation B from C:

$$\begin{array}{r} -10 = 9a + 3b + c \\ -(-2 = a - b + c) \\ \hline -8 = 8a + 4b \end{array}$$

STEP 3: Solve the equation from Step 1 in terms of a.

$$3 = 3a + 3b$$
$$3 - 3b = 3a$$
$$1 - b = a$$

STEP 4: Substitute the value for *a* into the equation from Step 2.

$-8 = 8(1 - b) + 4b$
$-8 = 8 - 4b$
$16 = 4b$
$4 = b$

STEP 5: Substitute the value for *b* into the equation from Step 3.

$a = 1 - (4)$
$a = -3$

STEP 6: Substitute the values for *a* and *b* into Equation *A*.

$1 = 4a + 2b + c$
$1 = 4(-3) + 2(4) + c$
$1 = -12 + 8 + c$
$1 = -4 + c$
$5 = c$

STEP 7: Substitute the values for *a*, *b*, and *c* into the standard form of a quadratic.

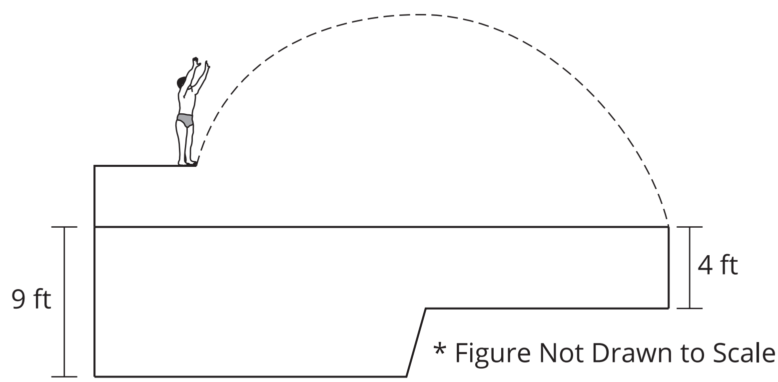
$y = -3x^2 + 4x + 5$

1. Use the Worked Example to write a quadratic equation that passes through the points (−1, 5), (0, 3), and (3, 9).

Happy Homes Development Company has hired Splish Splash Pools to create the community pool for their new development of homes. The rectangular pool is to have one section with a 4-foot depth, and another section with a 9-foot depth. The pool will also have a diving board. By law, the regulation depth of water necessary to have a diving board is 9 feet. Happy Homes would like to have the majority of the pool have a 4-foot depth in order to accommodate a large number of young children.

The diving board will be 3 feet above the edge of the pool's surface and extend 5 feet into the pool. After doing some research, Splish Splash Pools determined that the average diver would be 5 feet in the air when they are 8 feet from the edge of the pool, and 6 feet in the air when they are 10 feet from the edge of the pool.

2. Use the dive model shown to write an equation, and then determine the minimum length of the 9-foot-depth section of the pool. Explain your reasoning.



NOTES

Remember:

The general equation to represent height over time is $h(t) = -16t^2 + v_0t + h_0$ where v_0 is the initial velocity in feet per second and h_0 is the initial height in feet.

TALK the TALK

Fantastic Feats of Function

The Amazing Larry is a human cannonball. He would like to reach a maximum height of 30 feet during his next launch. Based on Amazing Larry's previous launches, his assistant DaJuan has estimated that this will occur when Larry is 40 feet from the cannon. When Amazing Larry is shot from the cannon, he is 10 feet above the ground.

1. Write a quadratic equation to represent Amazing Larry's height in terms of his distance.

Crazy Cornelius is a fire jumper. He is attempting to run and jump through a ring of fire. He runs for 10 feet. Then, he begins his jump just 4 feet from the fire and lands on the other side 3 feet from the fire ring. When Cornelius was 1 foot from the fire ring at the beginning of his jump, he was 3.5 feet in the air.

2. Write a quadratic equation to represent Crazy Cornelius' height in terms of his distance. Round to the nearest hundredth.

NOTES

Harsh Knarsh is attempting to jump across an alligator filled swamp. She takes off from a ramp 30 feet high with a speed of 95 feet per second.

3. Write a quadratic equation to represent Harsh Knarsh's height in terms of time.

Van McSlugger needs one more home run to advance to the next round of the home run derby. On the last pitch, he takes a swing and makes contact. Initially, he hits the ball at 5 feet above the ground. At 32 feet from home plate, his ball was 23.7 feet in the air, and at 220 feet from home plate, his ball was 70 feet in the air.

4. Consider the function that represents the relationship between the height of the ball and its distance from home plate.

a. If Van's ball needs to travel a distance of 399 feet in order to get the home run, did he succeed? Explain why or why not.

b. What was the maximum height of Van's baseball?

Quadratic Equations Cutouts

a. $f(x) = 2(x + 1)(x + 5)$	**b.** $f(x) = \frac{1}{3}x^2 + \pi x + 6.4$	**c.** $f(x) = -2.5(x - 3)(x - 3)$
d. $f(x) = (x - 1)^2$	**e.** $f(x) = 2(x - 1)(x - 5)$	**f.** $f(x) = x^2 + 12x - 1$
g. $f(x) = -(x + 4)^2 - 2$	**h.** $f(x) = -5x^2 - x + 21$	**i.** $f(x) = -(x + 2)^2 - 4$

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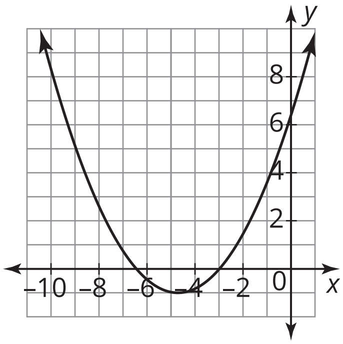
Graph Cutouts

A.



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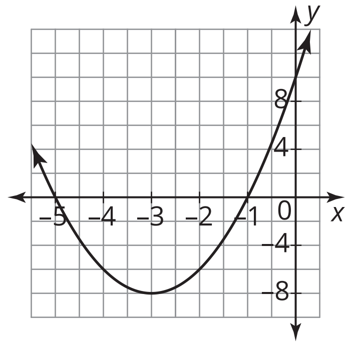

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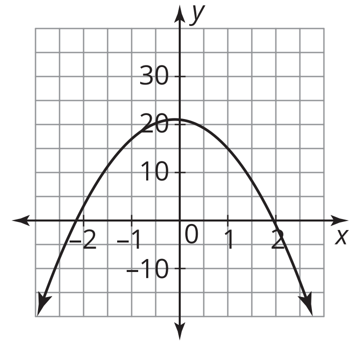

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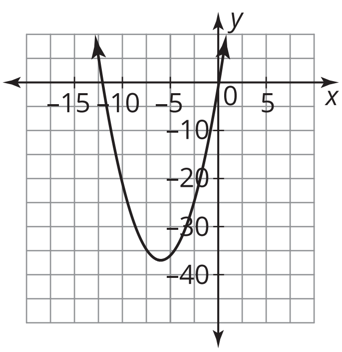

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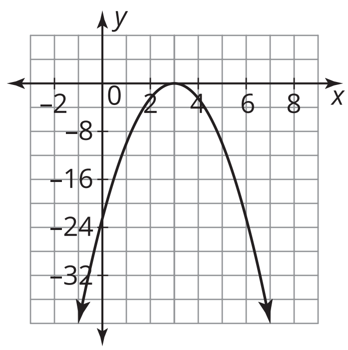

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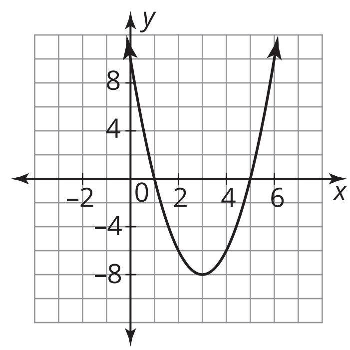
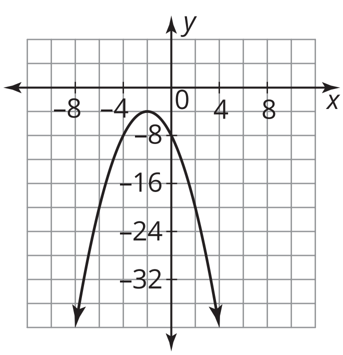
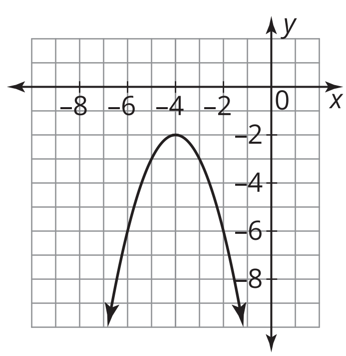


G.



H.



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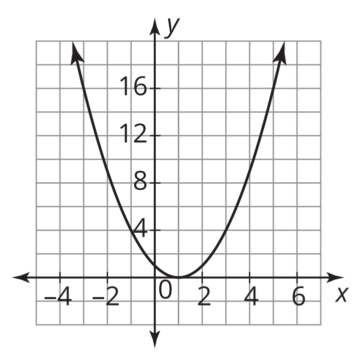

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<table>
<tr><th colspan="6">Standard Form
$f(x) = ax^2 + bx + c$, where $a \neq 0$</th></tr>
<tr><th colspan="6">Graphs and Their Functions</th></tr>
<tr><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><th colspan="6">Methods to Determine Key Characteristics</th></tr>
<tr><td colspan="2">Axis of Symmetry</td><td colspan="2">x-intercept(s)</td><td colspan="2">Concavity</td></tr>
<tr><td colspan="3">Vertex</td><td colspan="3">y-intercept</td></tr>
</table>

Factored Form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$		
Graphs and Their Functions		
Methods to Determine Key Characteristics		
Axis of Symmetry	***x*-intercept(s)**	**Concavity**

Vertex	***y*-intercept**

<table>
<tr><td colspan="6">Vertex Form
$f(x) = a(x - h)^2 + k$, where $a \neq 0$</td></tr>
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