4

True to Form

Forms of Quadratic Functions

MATERIALS

Tape or glue
Scissors

Lesson Overview

Students match quadratic equations with their graphs using key characteristics. The standard form, the factored form, and the vertex form of a quadratic equation are reviewed as is the concavity of a parabola. Students then sort each of the equations with their graphs according to the form in which the equation is written, identifying key characteristics of each function. Next, students analyze graphs of parabolas on numberless axes and choose possible functions that could model the graph. A Worked Example shows that a unique quadratic function is determined when the vertex and a point on the parabola are known, or the roots and a point on the parabola are known. Students are given information about a function and use it to determine the most efficient form to write the function. They then use the key characteristics of a graph and reference points to write a quadratic function, if possible. Finally, students analyze a Worked Example that demonstrates how to write and solve a system of equations to determine the unique quadratic function given three points on the graph. They then use this method to determine the quadratic function that models a problem situation and use it to answer a question about the situation.

Algebra 2
Systems of Equations and Inequalities

(3) The student applies mathematical processes to formulate systems of equations and inequalities, use a variety of methods to solve, and analyze reasonableness of solutions. The student is expected to:

(A) formulate systems of equations, including systems consisting of three linear equations in three variables and systems consisting of two equations, the first linear and the second quadratic.

(B) solve systems of three linear equations in three variables by using Gaussian elimination, technology with matrices, and substitution.

Quadratic and Square Root Functions, Equations, and Inequalities

(4) The student applies mathematical processes to understand that quadratic and square root functions, equations, and quadratic inequalities can be used to model situations, solve problems, and make predictions. The student is expected to:

(A) write the quadratic function given three specified points in the plane.

(D) transform a quadratic function $f(x) = ax^2 + bx + c$ to the form $f(x) = a(x - h)^2 + \mathrm{k}$ to identify the different attributes of $f(x)$.

Systems of Equations and Inequalities

(7) Number and algebraic methods. The student applies mathematical processes to simplify and perform operations on expressions and to solve equations. The student is expected to:

(B) add, subtract, and multiply polynomials.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 5.B, 5.F, 5.G

Essential Ideas

- The standard form of a quadratic function is written as $f(x) = ax^2 + bx + c$, where a does not equal 0.
- The factored form of a quadratic function is written as $f(x) = a(x - r_1)(x - r_2)$, where a does not equal 0.
- The vertex form of a quadratic function is written as $f(x) = a(x - h)^2 + k$, where a does not equal 0.
- The concavity of a parabola describes whether a parabola opens up or opens down. A parabola is concave down if it opens downward, and is concave up if it opens upward.
- A graphical method to determine a unique quadratic function involves using key points and the vertical distance between each point in comparison to the points on the basic function.
- An algebraic method to determine a unique quadratic function involves writing and solving a system of equations, given three reference points.

Lesson Structure and Pacing: 2 Days

Day 1

Engage

Getting Started: Be My One and Only

Students answer questions related to the number of points that determine a unique line and the number of points that determine a unique parabola.

Develop

Activity 4.1: Forms of Quadratic Functions

The standard form, factored form, and vertex form of a quadratic equation are reviewed, as well as the concavity of a parabola, and students identify the key characteristics that can be determined from a quadratic equation written in each form. Next, they match quadratic equations with their appropriate graphs using the vertex, x-intercepts, y-intercept, and a-value, depending on the form of the quadratic function. Students then sort each function and its graph according to the form in which the equation is written, and describe methods to identify key characteristics of the function given its form.

Activity 4.2: Modeling a Quadratic Graph

Given numberless graphs of parabolas, students identify possible equations that could model each graph. They then rewrite the equations for functions in different forms.

Day 2

Activity 4.3: Writing a Unique Quadratic Function

Students consider why you cannot create a unique quadratic function if you know only the vertex of the function. Worked Examples show that a unique quadratic equation can be determined when the vertex and a point on the parabola are known or when the roots and a point on the parabola are known. Students are given information about a function and use it to determine the most efficient form to write the function. They then write equations using given points, when possible.

Activity 4.4: Using Algebra to Write a Quadratic Equation

A Worked Example demonstrates how to write a unique quadratic equation given three non-linear points by writing and solving a system of equations. Linear combinations and substitution are used to determine the a-, b-, and c-values of the quadratic equation. Students solve similar real-world and mathematical problems.

Demonstrate

Talk the Talk: Fantastic Feats of Function

Students write quadratic equations to represent different situations. They identify three points on a parabola that models the situation and use those points to write a quadratic function.

ENGAGE

Getting Started: Be My One and Only

Facilitation Notes

In this activity, students answer questions related to the number of points that determine a unique line and the number of points that determine a unique parabola.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

Misconception

Students may draw lines through any two of the three non-collinear points. The question is asking whether a line can be drawn through all 3 points.

Questions to ask

- What are collinear points?
- Are all points collinear?
- What is an example of non-collinear points?
- Are any two points collinear?
- Are points *A*, *B*, and *C* collinear or non-collinear?
- How many lines are determined by 3 collinear points?
- How many lines are determined by 3 non-collinear points?
- How many parabolas can you draw through 2 points if one point is the vertex?
- Are all parabolas symmetric about some line?

Summary

A unique parabola can be drawn through three non-linear points.

DEVELOP

Activity 4.1
Forms of Quadratic Functions

Facilitation Notes

In this activity, the standard form, factored form, and vertex form of a quadratic equation are reviewed, as well as the concavity of a parabola, and students identify the key characteristics that can be determined from a quadratic equation written in each form. Next, they match quadratic equations with their appropriate graphs using the vertex, x-intercepts, y-intercept, and a-value, depending on the form of the quadratic function. Students then sort each function and its graph according to the form in which the equation is written, and describe methods to identify key characteristics of the function given its form.

Ask a student to read the introduction and definitions aloud. Discuss as a class.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Misconception

Students may reverse the concavity associated with the sign of *a*-values in the equation of the functions. Upward concavity is associated with positive *a*-values in the equation, and downward concavity is associated with negative *a*-values in the equation.

Questions to ask

- When a quadratic equation is written in standard form, what does the sign of the *a*-value indicate?
- Which feature of the quadratic equation determines whether the parabola has a maximum or minimum?
- When a quadratic equation is written in standard form, what is the significance of the *c*-value?
- When a quadratic equation is written in factored form, what does the sign of the *a*-value indicate?
- When a quadratic equation is written in factored form, what do the r_1 and r_2 values represent?
- When a quadratic equation is written in vertex form, what is the significance of (h, k)?
- When a quadratic function is written in vertex form, what does $x = h$ indicate?
- How can you determine whether a parabola passes through the origin?
- Christine's equation is written in which form?
- What information does Christine know from the form of her quadratic equation?
- Kate's equation is written in which form?
- What information does Kate know from the form of her quadratic equation?
- Which method required the use of the axis of symmetry and substitution?
- Who uses $x = -\frac{b}{2a}$ to determine the axis of symmetry?
- Who uses $x = \frac{r_1 + r_2}{2}$ to determine the axis of symmetry?

Have students work with a partner or in a group to complete Questions 3 and 4. Share responses as a class.

As students work, look for

Different methods to sort the equations and graphs before determining each corresponding set of equations and graphs. Is concavity considered? Are intercepts used?

Differentiation strategy

To scaffold support, suggest that students refer to the answers to Question 1 to complete the second half of each table located at the end of the activity.

Misconception

When the equation is written in factored form, students may use an incorrect sign for the x-intercepts. For example, if one factor is $(x - 2)$, students may use an x-value of -2 for the intercept rather than 2.

Questions to ask

- Did you sort the equations into separate piles before you taped them to the appropriate graph? If so, what determined those piles?
- Did you sort the graphs into two or more piles first? If so, what defined those piles?
- Did you separate the graphs by the orientation of the parabola?
- Did you identify all of the parabolas with a negative a-value and all of the parabolas with a positive a-value, then match them to the graphs?
- Did you use the y-intercepts to decide the most appropriate equation for the graph? Or did you use the y-intercepts to decide the most appropriate graph for the equation?
- In which form of the equation is the y-intercept most apparent?
- Did you use technology to help match the equation to the graph?
- Are x-intercepts helpful in some instances? What form was the equation written in?

Summary

The form of a quadratic function—standard, factored, or vertex—reveals different key characteristics, such as the x-intercept(s), y-intercept, vertex, axis of symmetry, and concavity up or down. For a quadratic function in any form, the a-value determines whether the parabola is concave up or down. When a quadratic function is in standard form, you can determine the y-intercept. When a quadratic function is in factored form, you can determine the x-intercepts. When a quadratic function is in vertex form, you can determine the coordinates of the vertex.

Activity 4.2
Modeling a Quadratic Graph

Facilitation Notes

In this activity, given numberless graphs of parabolas, students identify possible equations that could model each graph. They then rewrite the equations for functions in different forms.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

Misconception

Students may try to eliminate all but one equation, or attempt to choose the single most appropriate equation. Each graph can be described by more than one given equation.

Differentiation strategy

To scaffold support, suggest that students use graphing technology to verify their choices.

Questions to ask

- Are the zeros of the quadratic function real or imaginary?
- What graphic characteristic suggests real zeros? Imaginary zeros?
- Are the zeros of the quadratic function negative, positive, or both? What characteristic of the equation for the function gives you this information? What characteristics of the graph for the function gives you this information?
- Is the a-value of the quadratic function negative or positive? What graphic implications does this have?

Have students work with a partner or in a group to complete Question 2. Share responses as a class.

Questions to ask

- What characteristic indicates the form in which the given function is written?
- What algebraic properties are used to rewrite the quadratic function from factored form to standard form?
- What algebraic properties are used to rewrite the quadratic function from factored form to vertex form?
- What algebraic properties are used to rewrite the quadratic function from vertex form to standard form?
- Under what circumstances is completing the square a necessary process?
- Under what circumstances can the quadratic function not be written in factored form?

Summary

Key characteristics of a quadratic function—concavity, x-intercepts, y-intercept, and location of the vertex—can be identified using the equation. Rewriting the quadratic equation in a different form may be necessary.

Activity 4.3

Writing a Unique Quadratic Function

Facilitation Notes

In this activity, students consider why you cannot create a unique quadratic function if you know only the vertex of the function. Worked Examples show that a unique quadratic equation can be determined when the vertex and a point on the parabola are known or when the roots and a point on the parabola are known. Students are given information about a function, which they use to determine the most efficient form to write the function. They then write equations using given points, when possible.

Have students work with a partner or in a group to complete Question 1. Share the responses as a class.

As students work, look for

The use of vertex form of a quadratic equation and substitution of $h = 4$ and $k = 8$. Ask students how they determined the a-value for their equation.

Differentiation strategy

To scaffold support, write the different forms of a quadratic equation on the board for reference.

Misconception

Students may assume Pat's equation is correct and George's equation is incorrect because no a-value is specified. The a-value of $-\frac{1}{2}$ is obvious in George's equation, but because Pat chose an a-value of 1, it is not obvious in her equation.

Questions to ask

- George wrote his quadratic equation in which form?
- Pat wrote her quadratic equation in which form?
- Are George and Pat's equations equivalent?
- What a-value did George use?
- What a-value did Pat use?
- Do both quadratic equations have a vertex at (4, 8)?
- How many parabolas could have a vertex at (4, 8)?

Analyze and discuss the two Worked Examples as a class.

Questions to ask

- Is it possible that two or more quadratic functions have the same two roots?
- Does a specific vertex and two specified roots describe a unique function or many functions?
- Does the vertex and the *a*-value describe a unique function or many functions?
- Does a vertex and a point on the function determine one or many quadratic functions?
- If you are given the roots and a point on the function, is the function unique?
- Do the roots and the *a*-value describe a unique quadratic function?

Have students work with a partner or in a group to complete Question 2. Share the responses as a class.

As students work, look for

Connections made between the maximum or minimum point of a parabola and the vertex.

Questions to ask

- Is the minimum or maximum point of a quadratic function also the vertex of the function?
- If the *y*-values of two points are 0, are the points *x*- or *y*-intercepts?
- Under what circumstances is it easier to write the function in vertex form?
- If you know three points on the quadratic function, is it always easier to write the function in factored form?
- Under what circumstances is it easier to write the function in factored form?
- Under what circumstances is it easier to write the function in standard form?
- Is the maximum height of Max's baseball the location of the vertex of the parabola?

Have students work with a partner or in a group to complete Questions 3 and 4. Share the responses as a class.

Questions to ask

- How can the *a*-value be determined when writing a function for part (a)?
- What is the vertical distance between the vertex and a point on the basic function?
- How do you know the concavity of the parabola?
- Under what circumstances is the *a*-value negative?
- How can the *a*-value be determined when writing a function for part (b)?

- Is the a-value positive if the parabola opens upward?
- Why can't the axis of symmetry be determined in part (c)?
- How many points are needed to determine a unique quadratic function?

Differentiation strategy

To extend the activity, have students write each form of a quadratic function and summarize the minimum amount of information needed to write the equation in each form.

Summary

You can write a unique quadratic function given a vertex and the a-value or a point on the parabola. You can write a unique quadratic function given the roots and the a-value or a point on the parabola. The information you have about the parabola determines the most efficient form to use to write the function.

Activity 4.4

Using Algebra to Write a Quadratic Equation

Facilitation Notes

In this activity, a Worked Example demonstrates how to write a unique quadratic equation given three non-linear points by writing and solving a system of equations. Linear combinations and substitution are processes used to determine the a-, b-, and c-values of the quadratic equation. Students solve similar real-world and mathematical problems.

Ask a student to read the introduction aloud. Analyze and discuss the Worked Examples as a class.

Questions to ask

- How is Equation A determined?
- What point is used to write Equation B?
- What information is used to write Equation C?
- When Equation B is subtracted from Equation A, how do the signs of the terms in Equation B change? Do these same changes in signs occur when Equation B is subtracted from Equation C?
- What value is solved for first? Why do you suppose this was the case?
- Could another value be solved for first? Does it change the answer?

Have students work with a partner or in a group to complete Question 1. Share the responses as a class.

Questions to ask

- What is the first step in creating the system of equations?
- What is the quadratic equation written in standard form?
- How is the point (−1, 5) used to create the first equation?
- How is the point (0, 3) used to determine the second equation?
- What equation is written using the point (3, 9)?
- Did you substitute Equation 2 into Equation 1, or Equation 3? Does it make a difference?
- What variable did you solve for first?
- How is the value for c determined?
- What value is determined last?
- Is there more than one way to arrive at the correct quadratic function? If so, explain.

Have students work with a partner or in a group to read the scenario and complete Question 2. Share the responses as a class.

As students work, look for

The correct coordinates for the points to be (5, 3), (8, 5), and (10, 6). If students begin this problem by using incorrect points, the parts that follow will also be incorrect.

Differentiation strategy

To scaffold support, discuss a reasonable viewing window to graph the function equation $h(d) = -0.0\overline{33}x^2 + 1.1x - 1.\overline{66}$, Together, locate the point on the graph that represents the minimum length of the 9-foot-depth section of the pool.

Questions to ask

- What three points are used to create the system of equations?
- What are the coordinates of the point that represents the location of the diver on the diving board?
- What are the coordinates of the point at which the diver is 5 ft in the air and 8 ft from the edge of the pool?
- What are the coordinates of the point at which the diver is 6 ft in the air and 10 ft from the edge of the pool?
- How is the point (5, 3) used to create the first equation?
- How is the point (8, 5) used to determine the second equation?
- What equation is written using the point (10, 6)?
- What variable did you solve for first?
- What value is determined last?
- Is there more than one way to arrive at the correct quadratic function?

- What standard equation can be written using the values of *a*, *b*, and *c*?
- Did you use technology to graph the equation?
- Which point on the graph represents the minimum length of the 9-foot-depth section of the pool?

Summary

You can set up and solve systems of equations to determine the unique quadratic equation that models a parabola if you know three points on the parabola.

DEMONSTRATE

Talk the Talk: Fantastic Feats of Function

Facilitation Notes

In this activity, students write quadratic equations to represent different situations. They identify three points on a parabola that models the situation and use those points to write a quadratic function.

Have students work with a partner or in a group to complete Questions 1 through 3. Share the responses as a class.

As students work, look for

The use of the correct form of the quadratic equation in combination with the appropriate reference points.

Differentiation strategy

To scaffold support, suggest students sketch a diagram of each scenario and label key points.

Questions to ask

- Which key characteristic is associated with the given information?
- If Amazing Larry reaches a maximum height of 30 feet, will this be the vertex of the graph?
- In which form did you write your quadratic function?
- If the cannon is 10 feet above the ground, is this associated with the *y*-intercept of the graph?
- What is the concavity of the parabola? How do you know?
- Is the *a*-value of the function a positive number or a negative number?
- What distance does Crazy Cornelius travel before he leaves the ground?
- What distance does Crazy Cornelius travel as he lands back on the ground?
- Are the points or distances at which Crazy Cornelius lifts off the ground and lands back down on the ground associated with the roots of the function, or *x*-intercepts?

- What distance does Crazy Cornelius travel when he reaches a height of 3.5 feet?
- If Harsh Knarsh begins on a ramp 30 feet high, is this associated with the *y*-intercept of the graph?
- What function or equation is associated with vertical motion?
- Is the equation for vertical motion written in standard form?
- What is the initial height in this situation?
- What is the initial velocity in this situation?

Have students work with a partner or in a group to complete Question 4. Share the responses as a class.

Questions to ask

- Is this a vertical motion situation?
- How can you write the function for the height of the ball in terms of its horizontal distance?
- What reference points are used in this situation?
- How can graphing technology be used to determine the total distance the ball traveled?
- What is the height of the ball when it hits the ground?
- If Van's ball traveled a total of 430.4 feet, did he succeed?
- Where on the graph is the maximum height of the baseball?
- How can graphing technology be used to determine the maximum height of Van's baseball?

Summary

If you can determine key points on a parabola that models a real-world context, you can use algebra or graphing technology to write the quadratic function that models the situation.

NOTES

True to Form

Forms of Quadratic Functions

4

Warm Up

Identify the axis of symmetry of each quadratic function.

1. $f(x) = -4(x - 3)^2 + 2$

2. $g(x) = 2(x + 5)(x - 7)$

3. $h(x) = 4x^2 + 6x + 1$

Learning Goals

- Determine how many points are necessary to create a unique quadratic equation.
- Match a quadratic function with its corresponding graph.
- Identify key characteristics of quadratic functions based on the form of the function.
- Analyze the different forms of quadratic functions.
- Use key characteristics of specific forms of quadratic functions to write equations.
- Derive a quadratic equation given three points using a system of equations.
- Write quadratic functions to represent problem situations.

Key Terms

- standard form of a quadratic function
- factored form of a quadratic function
- vertex form of a quadratic function
- concavity of a parabola

You have explored the key characteristics of different forms of quadratic functions. How can you use these characteristics to write a quadratic equation to model the graph of a parabola, even if you only know certain points on the graph?

LESSON 4: True to Form • 1

Warm Up Answers

1. $x = 3$
2. $x = 1$
3. $x = -\frac{3}{4}$

Answers

1a. I can draw an infinite number of lines through point *A*.

1b. I can draw exactly one line through points *A* and *B*.

1c. I cannot draw any lines through all points *A*, *B*, and *C*.

2. I need two points to draw a unique line.

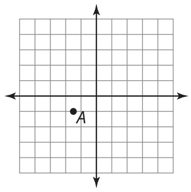
GETTING STARTED

Be My One and Only

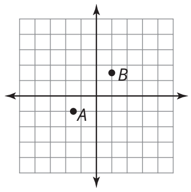
Consider the family of linear functions.

1. Use the given point(s) to sketch possible solutions.

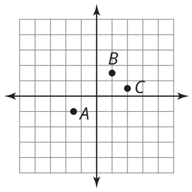
a. How many lines can you draw through point *A*?



b. How many lines can you draw through both points *A* and *B*?



c. How many lines can you draw through all points *A*, *B*, and *C*?



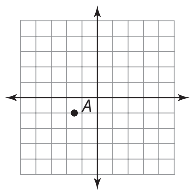
2. What is the minimum number of points you need to draw a unique line?

2 • TOPIC 2: Exploring and Analyzing Patterns

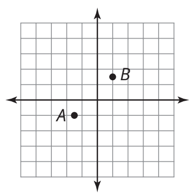
Consider the family of quadratic functions.

3. Use the given point(s) to sketch possible solutions.

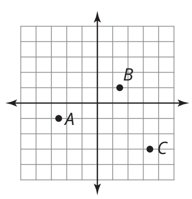
a. How many parabolas can you draw through point *A*?



b. How many parabolas can you draw through both points *A* and *B*?



c. How many parabolas can you draw through all points *A*, *B*, and *C*?



Answer

3a. I can draw an infinite number of parabolas through point *A*.

3b. I can draw an infinite number of parabolas through points *A* and *B*.

3c. I can draw one parabola through points *A*, *B*, and *C*.

Answer

1a. Standard form:
I can determine the y-intercept and whether the parabola is concave up or concave down when the quadratic function is in standard form.

1b. Factored form:
I can determine the x-intercepts and whether the parabola is concave up or concave down when the quadratic function is in factored form.

1c. Vertex form:
I can determine the vertex, whether the parabola is concave up or concave down, and the axis of symmetry when the quadratic function is in vertex form.

NOTES

ACTIVITY 4.1

Forms of Quadratic Functions

Recall that quadratic functions can be written in different forms.

Standard form: $f(x) = ax^2 + bx + c$, where a does not equal 0
Factored form: $f(x) = a(x - r_1)(x - r_2)$, where a does not equal 0
Vertex form: $f(x) = a(x - h)^2 + k$, where a does not equal 0

The graphs of quadratic functions can be described using key characteristics: x-intercept(s), y-intercept, vertex, axis of symmetry, and concavity.

Concavity of a parabola describes whether a parabola opens up or opens down. A parabola is concave up if it opens upward; a parabola is concave down if it opens downward.

1. The form of a quadratic function reveals different key characteristics. State the characteristics you can determine from each form.

a. Standard form

b. Factored form

C. Vertex form

2. **Christine and Kate were asked to determine the vertex of two different quadratic functions, each written in a different form. Analyze their calculations.**

Christine

$f(x) = 2x^2 + 12x + 10$

The quadratic function is in standard form. So I know the axis of symmetry is $x = \frac{-b}{2a}$.

$$x = \frac{-12}{2(2)}$$
$$= -3$$

Now that I know the axis of symmetry, I can substitute that value into the function to determine the y-coordinate of the vertex.

$$f(-3) = 2(-3)^2 + 12(-3) + 10$$
$$= 2(9) - 36 + 10$$
$$= 18 - 36 + 10$$
$$= -8$$

Therefore, the vertex is $(-3, -8)$.

Kate

$g(x) = \frac{1}{2}(x + 3)(x - 7)$

The form of the function tells me the x-intercepts are -3 and 7. I also know the x-coordinate of the vertex is directly in the middle of the x-intercepts. So, all I have to do is calculate their average.

$$x = \frac{-3 + 7}{2}$$
$$= \frac{4}{2} = 2$$

Now that I know the x-coordinate of the vertex, I can substitute that value into the function to determine the y-coordinate.

$$g(2) = \frac{1}{2}(2 + 3)(2 - 7)$$
$$= \frac{1}{2}(5)(-5)$$
$$= -12.5$$

Therefore, the vertex is $(2, -12.5)$.

a. **How are these methods similar? How are they different?**

b. **What must Kate do to use Christine's method?**

c. **What must Christine do to use Kate's method?**

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Answers

2a. Both methods require that you determine the axis of symmetry, and then substitute that value into the function to determine the y-coordinate of the vertex. The methods are different in the way the axis of symmetry is determined. Christine used $x = -\frac{b}{2a}$ and Kate used $x = \frac{(r_1 + r_2)}{2}$.

2b. Kate knows the a-value from the form of her quadratic equation. She must multiply the factors together and combine like terms. She would then have a quadratic function in standard form and can determine the b-value.

2c. Christine must factor the quadratic function or use the Quadratic Formula to determine the x-intercepts. Once she determines the x-intercepts, she can use the same method as Kate.

Answers

3a. Graph A, Equation b
Graph B, Equation a
Graph C, Equation h
Graph D, Equation f
Graph E, Equation c
Graph F, Equation e
Graph G, Equation i
Graph H, Equation g
Graph I, Equation d

3b. Answers will vary. Students may identify the graphs by their vertex, x-intercept(s), y-intercept, and a-value depending on the form of the quadratic function. They may also substitute values of points into the functions or make a table.

4. See tables located at the end of the lesson.

Do not use technology for this activity. Instead, use the information you can determine by analyzing each equation and each graph to determine a match.

3. Cut out each quadratic equation and graph located at the end of the lesson.

a. Tape each quadratic equation to its corresponding graph.

b. Explain the method(s) you used to match each equation with its graph.

4. Analyze the three tables located at the end of the lesson. Tape each function and its corresponding graph from Question 3 in the "Graphs and Their Functions" section of the appropriate table. Then, explain how you can determine each key characteristic based on the form of the given function.

ACTIVITY

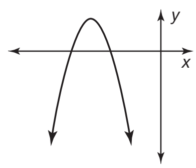
4.2 Modeling a Quadratic Graph

1. Analyze each graph. Then, circle the function(s) which could model the graph. Describe the reasoning you used to either eliminate or choose each function.

Think about:

What information is given by each function and the relative position of its graph?

a.



$f_1(x) = -2(x + 1)(x + 4)$

$f_2(x) = -\frac{1}{3}x^2 - 3x - 6$

$f_3(x) = 2(x + 1)(x + 4)$

$f_4(x) = 2x^2 - 8.9$

$f_5(x) = 2(x - 1)(x - 4)$

$f_6(x) = -(x - 6)^2 + 3$

$f_7(x) = -3(x + 2)(x - 3)$

$f_8(x) = -(x + 6)^2 + 3$

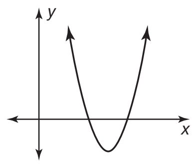
Answers

1a. The function f_1 is a possibility because it has a negative a-value and 2 negative x-intercepts.
The function f_2 is a possibility because it has a negative a-value and a negative y-intercept.
The function f_8 is a possibility because it has a negative a-value and a vertex in Quadrant II.

Answers

1b. The function f_1 is a possibility because it has a positive a-value and a vertex in Quadrant IV.
The function f_3 is a possibility because it has a positive a-value and a positive y-intercept.

b.



$f_1(x) = 2(x - 75)^2 - 92$

$f_2(x) = (x - 8)(x + 2)$

$f_3(x) = 8x^2 - 88x + 240$

$f_4(x) = -3(x - 1)(x - 5)$

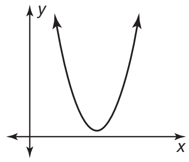
$f_5(x) = -2(x - 75)^2 - 92$

$f_6(x) = x^2 + 6x - 2$

$f_7(x) = 2(x + 4)^2 - 2$

$f_8(x) = (x + 1)(x + 3)$

c.



$f_1(x) = 3(x + 1)(x - 5)$ $f_2(x) = 2(x + 6)^2 - 5$ $f_3(x) = 4x^2 - 400x + 10{,}010$

$f_4(x) = 3(x + 1)(x + 5)$ $f_5(x) = 2(x - 6)^2 + 5$ $f_6(x) = x^2 + 2x - 5$

2. Consider the two functions shown from Question 1. Identify the form of the function given, and then write the function in the other two forms, if possible. If it is not possible, explain why.

a. $f_1(x) = -2(x + 1)(x + 4)$ **b.** $f_5(x) = 2(x - 6)^2 + 5$

Answers

1c. The function f_3 is a possibility because it has a positive a-value and a positive y-intercept.
The function f_5 is a possibility because it has a positive a-value and a vertex in Quadrant I.

2a. The function is given in factored form.
standard form:

$f_1(x) = -2(x^2 + 5x + 4)$

$= -2x^2 - 10x - 8$

vertex form:

$f_1(x) = -2(x^2 + 5x + 4)$

$= -2\left(x^2 + 5x + \frac{25}{4}\right) + \frac{9}{2}$

$= -2\left(x + \frac{5}{2}\right)^2 + \frac{9}{2}$

2b. The function is given in vertex form.
standard form:

$f_5(x) =$
$2(x^2 - 12x + 36) + 5$

$= 2x^2 - 24x + 72 + 5$

$= 2x^2 - 24x + 77$

factored form:
Sample answer.
The function does not cross the x-axis, therefore it does not have real number x-intercepts. I cannot factor this function.

Answer

1. Both George and Pat are correct.
George and Pat each used the vertex form of a quadratic equation and substituted $h = 4$ and $k = 8$. George chose $a = -\frac{1}{2}$, and Pat chose $a = 1$.
There was not enough information given to create a unique quadratic equation, therefore, both equations represent a quadratic equation with the vertex (4, 8).

ACTIVITY **4.3**

Writing a Unique Quadratic Function

You have used properties of linear functions to write linear equations. In this activity, you will use properties of quadratic functions to write quadratic equations in various forms.

1. George and Pat each wrote a quadratic equation with a vertex of (4, 8). Analyze each student's work. Describe the similarities and differences in their equations and determine who is correct.

George

$y = a(x - h)^2 + k$

$y = a(x - 4)^2 + 8$

$y = -\frac{1}{2}(x - 4)^2 + 8$

Pat

$y = a(x - h)^2 + k$

$y = a(x - 4)^2 + 8$

$y = (x - 4)^2 + 8$

You can write a unique quadratic function given a vertex and a point on the parabola.

Worked Example

Write the quadratic function given the vertex (5, 2) and the point (4, 9).

Substitute the given values into the vertex form of the function.

$f(x) = a(x - h)^2 + k$

$9 = a(4 - 5)^2 + 2$

Then solve for a.

$9 = a(-1)^2 + 2$

$9 = 1a + 2$

$7 = 1a$

$7 = a$

Finally, substitute the a-value into the function.

$f(x) = 7(x - 5)^2 + 2$

You can write a unique quadratic function given the roots and a point on the parabola.

Worked Example

Write a quadratic function given the roots (−2, 0) and (4, 0), and the point (1, 6).

Substitute the given values into the factored form of the function.

$$f(x) = a(x - r_1)(x - r_2)$$
$$6 = a(1 - (-2))(1 - 4)$$

Then solve for a.

$$6 = a(1 + 2)(1 - 4)$$
$$6 = a(3)(-3)$$
$$6 = -9a$$
$$-\frac{2}{3} = a$$

Finally, substitute the a-value into the function.

$$f(x) = -\frac{2}{3}(x + 2)(x - 4)$$

2. Determine which form of a quadratic function would be most efficient to write the function using the given information. Write *standard form, factored form, vertex form,* or *none* in the space provided.

a. minimum point (6, −75)
y-intercept (0, 15) ____________________

b. points (2, 0), (8, 0), and (4, 6) ____________________

c. points (100, 75), (450, 75), and (150, 95) ____________________

d. points (3, 3), (4, 3), and (5, 3) ____________________

e. x-intercepts (7.9, 0) and (−7.9, 0)
point (−4, −4) ____________________

f. roots (3, 0) and (12, 0)
point (10, 2) ____________________

g. Max hits a baseball off a tee that is 3 feet high. The ball reaches a maximum height of 20 feet when it is 15 feet from the tee. ____________________

Answer

2a. vertex form
2b. factored form
2c. standard form
2d. none
2e. factored form
2f. factored form
2g. vertex form

Answer

3a. $f(x) = -3(x + 3)^2 + 4$
Point $(-4, 1)$ is one unit from the axis of symmetry. The vertical distance between the vertex and $(-4, 1)$ is 3. I know the parabola opens down, therefore the a-value must be -3.

3b. $f(x) = 2(x - 3)^2 - 2$
Point $(4, 0)$ is one unit from the axis of symmetry. The vertical distance between the vertex and $(4, 0)$ is 2, therefore the a-value must be 2.

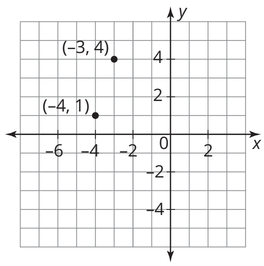
3c. I cannot determine the equation of the unique parabola using the given points. I do not know the axis of symmetry; therefore, I cannot determine the relationship between the points plotted and the reference points on the basic function.

4. Wilhemina is correct only if it is given that one of the two points is the vertex. Otherwise, you would need a minimum of three points to write a unique quadratic function.

3. Write a quadratic function that includes the given points. If it is not possible to write a function, state why not.

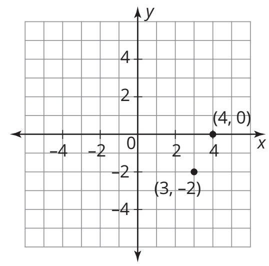
a. Given: vertex $(-3, 4)$; point $(-4, 1)$

$f(x)$ = ____________________



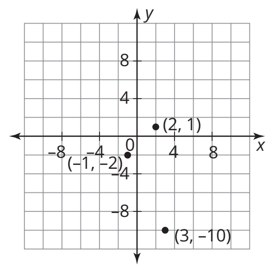
b. Given: vertex $(3, -2)$; one of two x-intercepts $(4, 0)$

$f(x)$ = ____________________



c. Given: points $(2, 1)$, $(-1, -2)$, $(3, -10)$

$f(x)$ = ____________________



4. Wilhemina says that she can write a unique quadratic function given only two points. Is she correct? Explain your reasoning.

ACTIVITY

4.4

Using Algebra to Write a Quadratic Equation

In the previous activity, there were times when you could not determine a quadratic equation using known strategies. You know how to use technology to determine the equation, but what if you don't have technology?

You know that you need a minimum of 3 non-linear points to create a unique parabola. To create an equation to represent the parabola, you can use a system of equations.

Worked Example

Consider the three points given in Question 3 part (c) in the previous activity: A (2, 1), B (−1, −2), and C (3, −10).

To write an equation in standard form to represent a parabola that passes through three given points, begin by substituting the x- and y-values of each point into $y = ax^2 + bx + c$.

Point A: $1 = a(2)^2 + b(2) + c$
$1 = 4a + 2b + c$ — Equation A: $1 = 4a + 2b + c$

Point B: $-2 = a(-1)^2 + b(-1) + c$
$-2 = a - b + c$ — Equation B: $-2 = a - b + c$

Point C: $-10 = a(3)^2 + b(3) + c$
$-10 = 9a + 3b + c$ — Equation C: $-10 = 9a + 3b + c$

Now, use linear combinations and substitution to solve for a, b, and c.

STEP 1: Subtract Equation B from A:

$$\begin{array}{r} 1 = 4a + 2b + c \\ -(-2 = a - b + c) \\ \hline 3 = 3a + 3b \end{array}$$

STEP 2: Subtract Equation B from C:

$$\begin{array}{r} -10 = 9a + 3b + c \\ -(-2 = a - b + c) \\ \hline -8 = 8a + 4b \end{array}$$

STEP 3: Solve the equation from Step 1 in terms of a.

$$3 = 3a + 3b$$
$$3 - 3b = 3a$$
$$1 - b = a$$

Answers

1. Write the equation in standard form for each point that you know.

   Equation 1: $5 = a - b + c$

   Equation 2: $3 = c$

   Equation 3: $9 = 9a + 3b + c$

   Substitute Equation 2 into Equation 1 and solve for a.

   $5 = a - b + 3$

   $2 = a - b$

   $a = b + 2$

   Substitute the value for a in terms of b and your value for c into Equation 3.

   $9 = 9(b + 2) + 3b + 3$

   $9 = 9b + 18 + 3b + 3$

   $9 = 12b + 21$

   $-12 = 12b$

   $-1 = b$

   Substitute the values for b and c into Equation 1.

   $5 = a - (-1) + 3$

   $5 = a + 4$

   $a = 1$

   Substitute the values of a, b, and c into a quadratic equation in standard form.

   $f(x) = x^2 - x + 3$

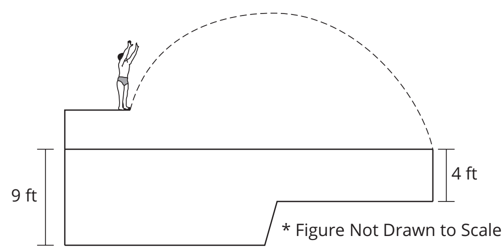
STEP 4: Substitute the value for a into the equation from Step 2.	$-8 = 8(1 - b) + 4b$ $-8 = 8 - 4b$ $16 = 4b$ $4 = b$
STEP 5: Substitute the value for b into the equation from Step 3.	$a = 1 - (4)$ $a = -3$
STEP 6: Substitute the values for a and b into Equation A.	$1 = 4a + 2b + c$ $1 = 4(-3) + 2(4) + c$ $1 = -12 + 8 + c$ $1 = -4 + c$ $5 = c$
STEP 7: Substitute the values for a, b, and c into the standard form of a quadratic.	$y = -3x^2 + 4x + 5$

1. Use the Worked Example to write a quadratic equation that passes through the points (−1, 5), (0, 3), and (3, 9).

Happy Homes Development Company has hired Splish Splash Pools to create the community pool for their new development of homes. The rectangular pool is to have one section with a 4-foot depth, and another section with a 9-foot depth. The pool will also have a diving board. By law, the regulation depth of water necessary to have a diving board is 9 feet. Happy Homes would like to have the majority of the pool have a 4-foot depth in order to accommodate a large number of young children.

The diving board will be 3 feet above the edge of the pool's surface and extend 5 feet into the pool. After doing some research, Splish Splash Pools determined that the average diver would be 5 feet in the air when they are 8 feet from the edge of the pool, and 6 feet in the air when they are 10 feet from the edge of the pool.

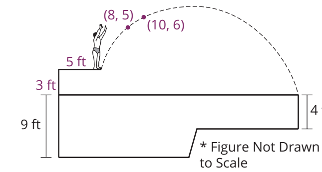
2. Use the dive model shown to write an equation, and then determine the minimum length of the 9-foot-depth section of the pool. Explain your reasoning.



The minimum length of the 9-foot-depth section of the pool is 31.41 feet. I graphed $h(d)$ using technology and determined the x-intercepts.

Answers

2.



Equation 1:
$3 = 25a + 5b + c$

Equation 2:
$5 = 64a + 8b + c$

Equation 3:
$6 = 100a + 10b + c$

Subtract Equation 2 from Equation 1 to get $-2 = -39a - 3b$.

Subtract Equation 3 from Equation 2 to get $-1 = -36a - 2b$.

Solve the equation $-1 = -36a - 2b$ in terms of b:
$b = -18a + 0.5$

Substitute the value of b into the equation you found in the first step.

$-2 = -39a - 3(-18a + 0.5)$

$a = -0.0\overline{33}$

Substitute the value of a into the equation you found when solving for b.

$b = -18(-0.0\overline{33}) + 0.5$

$b = 1.1$

Substitute the values of a and b into any of the original equations and solve for c.

$6 = 100(-0.0\overline{33}) + 10(1.1) + c$

$c = -1.\overline{66}$

Substitute the values of a, b, and c into a quadratic equation in standard form.

$h(d) = -0.0\overline{33}x^2 + 1.1x - 1.\overline{66}$

Answers

1. Let $h(d)$ represent Amazing Larry's height in terms of the distance, d.

$h(d) = a(d - 40)^2 + 30$

$10 = a(0 - 40)^2 + 30$

$10 = 1600a + 30$

$-\frac{1}{80} = a$

$h(d) = -\frac{1}{80}(d - 40)^2 + 30$

2. Let $h(d)$ represent Crazy Cornelius's height in terms of the distance, d.

$h(d) = a(d - r_1)(d - r_2)$

$3.5 = a(13 - 10)(13 - 17)$

$3.5 = a(3)(-4)$

$3.5 = -12a$

$-0.29 = a$

$h(d) = -0.29(d - 10)(d - 17)$

NOTES

Remember:

The general equation to represent height over time is $h(t) = -16t^2 + v_0t + h_0$ where v_0 is the initial velocity in feet per second and h_0 is the initial height in feet.

TALK the TALK

Fantastic Feats of Function

The Amazing Larry is a human cannonball. He would like to reach a maximum height of 30 feet during his next launch. Based on Amazing Larry's previous launches, his assistant DaJuan has estimated that this will occur when Larry is 40 feet from the cannon. When Amazing Larry is shot from the cannon, he is 10 feet above the ground.

1. Write a quadratic equation to represent Amazing Larry's height in terms of his distance.

Crazy Cornelius is a fire jumper. He is attempting to run and jump through a ring of fire. He runs for 10 feet. Then, he begins his jump just 4 feet from the fire and lands on the other side 3 feet from the fire ring. When Cornelius was 1 foot from the fire ring at the beginning of his jump, he was 3.5 feet in the air.

2. Write a quadratic equation to represent Crazy Cornelius' height in terms of his distance. Round to the nearest hundredth.

ELL Tip

Read aloud the title of the activity "Fantastic *Feats* of Function." Assess students' prior knowledge of the term *feat*. Define a *feat* as *an achievement that requires great skill, courage, or strength*. In the context of the section title, discuss how a "function *feat*" could refer to the *specific characteristics of a quadratic function for each scenario in the activity*. Create a list of synonyms for *feat*, such as *accomplishment, attainment, and triumph.*

NOTES

Harsh Knarsh is attempting to jump across an alligator filled swamp. She takes off from a ramp 30 feet high with a speed of 95 feet per second.

3. Write a quadratic equation to represent Harsh Knarsh's height in terms of time.

Van McSlugger needs one more home run to advance to the next round of the home run derby. On the last pitch, he takes a swing and makes contact. Initially, he hits the ball at 5 feet above the ground. At 32 feet from home plate, his ball was 23.7 feet in the air, and at 220 feet from home plate, his ball was 70 feet in the air.

4. Consider the function that represents the relationship between the height of the ball and its distance from home plate.

a. If Van's ball needs to travel a distance of 399 feet in order to get the home run, did he succeed? Explain why or why not.

b. What was the maximum height of Van's baseball?

Answers

3. Let $h(t)$ represent Harsh Knarsh's height in terms of the time, t.

   $h(t) = -16t^2 + v_0t + h_0$

   $h(t) = -16t^2 + 95t + 30$

4a. Van did succeed. Van's ball went 430.4 feet.
Sample answer.

I determined the function for the height of the ball in terms of its horizontal distance to be $h(d) = -0.0015d^2 + 0.634d + 5$. I used graphing technology to graph the function and then determined the distance of his ball when the height was 0.

4b. The maximum height of Van's ball was about 72 feet.

ELL Tip

Determine whether students are familiar with the term *home run derby*. If not, define a *derby as a race or contest open to a specific category of contestants.* In the context of Question 4, a *home run derby is a contest to see which person can hit the most number of home runs in a specified amount of time.* Read aloud Question 4 and clarify any further misunderstandings students may have about the term *home run derby.*

Quadratic Equations Cutouts

a. $f(x) = 2(x + 1)(x + 5)$	**b.** $f(x) = \frac{1}{3}x^2 + \pi x + 6.4$	**c.** $f(x) = -2.5(x - 3)(x - 3)$
d. $f(x) = (x - 1)^2$	**e.** $f(x) = 2(x - 1)(x - 5)$	**f.** $f(x) = x^2 + 12x - 1$
g. $f(x) = -(x + 4)^2 - 2$	**h.** $f(x) = -5x^2 - x + 21$	**i.** $f(x) = -(x + 2)^2 - 4$

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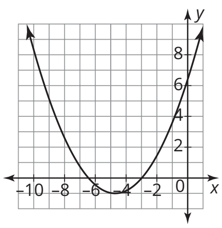
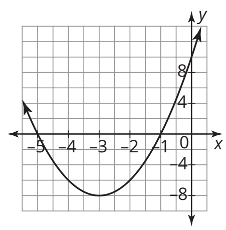
Graph Cutouts

A.



B.



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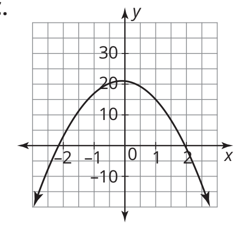

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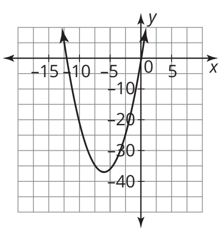
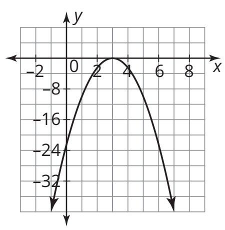
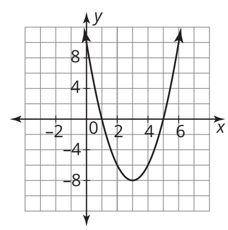


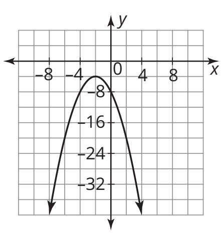
E.



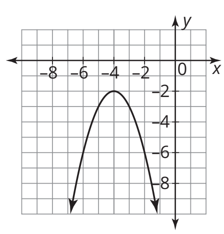
F.



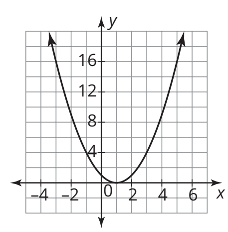
G.



H.



I.



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<table>
<tr><th colspan="6">Standard Form
$f(x) = ax^2 + bx + c$, where $a \neq 0$</th></tr>
<tr><th colspan="6">Graphs and Their Functions</th></tr>
<tr><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><th colspan="6">Methods to Determine Key Characteristics</th></tr>
<tr><td colspan="2">Axis of Symmetry</td><td colspan="2">x-intercept(s)</td><td colspan="2">Concavity</td></tr>
<tr><td colspan="3">Vertex</td><td colspan="3">y-intercept</td></tr>
</table>

Concave down when $a < 0$.

Vertex: Use $\frac{(-b)}{2a}$ to determine the x-coordinate of the vertex. Then substitute that value into the equation and solve for y.

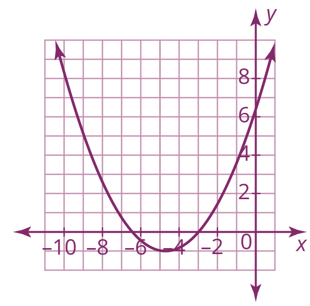
y-intercept: c-value

Answers

4. Standard Form

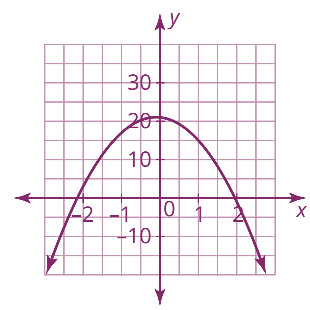
Graphs:

A.



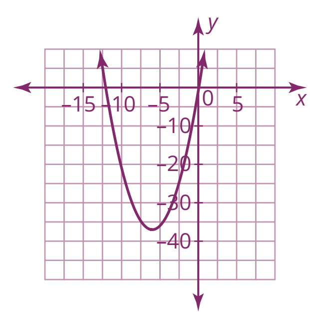
b. $f(x) = \frac{1}{3}x^2 + \pi x + 6.4$

C.



h. $f(x) = -5x^2 - x + 21$

D.



f. $f(x) = x^2 + 12x - 1$

Axis of symmetry: $x = -\frac{b}{2a}$

x-intercept(s): Substitute 0 for y, and then solve for x using the Quadratic Formula, factoring, or graphing technology.

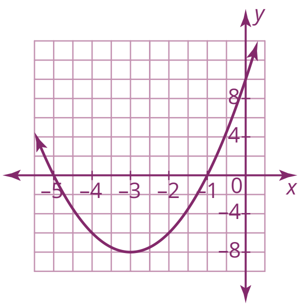
Concavity: Concave up when $a > 0$.

Answers

Factored Form

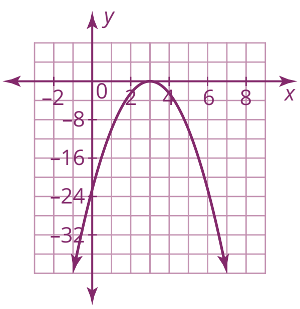
Graphs:

B.



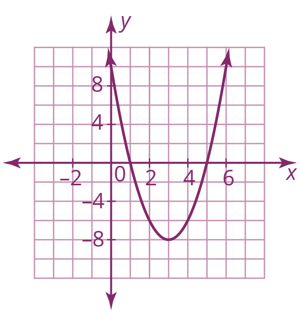
a. $f(x) = 2(x + 1)(x + 5)$

E.



c. $f(x) = -2.5(x - 3)(x - 3)$

F.



e. $f(x) = 2(x - 1)(x - 5)$

Axis of symmetry: $x = \frac{(r_1 + r_2)}{2}$

x-intercept(s): $(r_1, 0), (r_2, 0)$

Concavity: Concave up when $a > 0$.

Concave down when $a < 0$.

Factored Form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$		
Graphs and Their Functions		
Methods to Determine Key Characteristics		
Axis of Symmetry	***x*-intercept(s)**	**Concavity**
Vertex	***y*-intercept**	

Vertex: Use $\frac{(r_1 + r_2)}{2}$ to determine the x-coordinate of the vertex. Then substitute that value into the equation and solve for y.

y-intercept: Substitute 0 for x, and then solve for y.

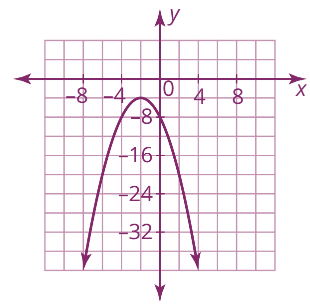
<table>
<tr><th colspan="6">Vertex Form
$f(x) = a(x - h)^2 + k$, where $a \neq 0$</th></tr>
<tr><th colspan="6">Graphs and Their Functions</th></tr>
<tr><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><th colspan="6">Methods to Determine Key Characteristics</th></tr>
<tr><td colspan="2">Axis of Symmetry</td><td colspan="2">x-intercept(s)</td><td colspan="2">Concavity</td></tr>
<tr><td colspan="3">Vertex</td><td colspan="3">y-intercept</td></tr>
</table>

Answers

Vertex Form

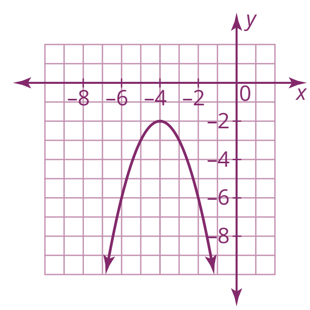
Graphs:

G.



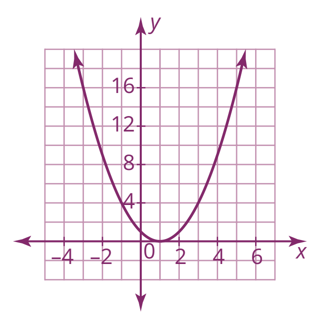
i. $f(x) = -(x + 2)^2 - 4$

H.



g. $f(x) = -(x + 4)^2 - 2$

I.



d. $f(x) = (x - 1)^2$

Axis of symmetry: $x = h$

x-intercept(s): Substitute 0 for y, and then solve for x using the Quadratic Formula, factoring, or graphing technology.

Concavity: Concave up when $a > 0$. Concave down when $a < 0$.

Vertex: (h, k)

y-intercept: Substitute 0 for x, and then solve for y.