5

The Root of the Problem

Solving Quadratic Equations

MATERIALS

None

Lesson Overview

Students solve quadratic equations of the form $y = ax^2 + bx + c$. They first factor trinomials and use the Zero Product Property. Students then complete the square to determine the roots of a quadratic equation that cannot be factored. Finally, students use the Quadratic Formula to solve problems in and out of context.

Algebra 2

Quadratic and Square Root Functions, Equations, and Inequalities

(4) The student applies mathematical processes to understand that quadratic and square root functions, equations, and quadratic inequalities can be used to model situations, solve problems, and make predictions. The student is expected to:

(F) solve quadratic and square root equations.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 5.B, 5.F, 5.G

Essential Ideas

- One method of solving quadratic equations in the form $0 = ax^2 + bx + c$ is to factor the trinomial expression and use the Zero Product Property.
- When a quadratic equation in the form $0 = ax^2 + bx + c$ is not factorable, completing the square is an alternative method of solving of the equation.
- The Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 4ac}}{2a}$, can be used to solve to any quadratic equation written in general form, $0 = ax^2 + bx + c$, where a, b, and c represent real numbers and $a \ne 0$.
- A system of equations containing two quadratic equations can be solved algebraically and graphically.
- The Quadratic Formula, substitution, and factoring are used to algebraically solve systems of equations.

Lesson Structure and Pacing: 2 Days

Day 1

Engage

Getting Started: Grassroots

Students use the Properties of Equality to solve quadratic equations written in specific forms. They analyze a strategy for solving an equation in the form $y = x^2 + d$ that uses factoring.

Develop

Activity 5.1: Solving Quadratic Equations by Factoring

A Worked Example shows how to use factoring and the Zero Product Property to determine the roots of a given quadratic equation. Students then practice solving quadratic equations by factoring.

Activity 5.2: Completing the Square to Determine Roots

Students analyze a Worked Example that demonstrates how to complete the square to determine the zeros of the quadratic function that is unfactorable. Students then complete the square to determine the roots of other quadratic equations both in and out of context.

Day 2

Activity 5.3: Using the Quadratic Formula

Students review the Quadratic Formula as a method to calculate the roots of any quadratic equation written in general form. They analyze a Worked Example demonstrating how to use the Quadratic Formula to calculate the roots of a quadratic equation. They then use the formula to solve a problem in context. Students identify and correct a common error that occurs when using the Quadratic Formula and then use the Quadratic Formula to calculate the zeros of other functions.

Demonstrate

Talk the Talk: Show Me the Ways

Students determine the roots of a quadratic equation using four methods: factoring, completing the square, using the Quadratic Formula, and graphing.

Facilitation Notes

In this activity, students use the Properties of Equality to solve quadratic equations written in specific forms. They analyze a strategy for solving an equation in the form $y = x^2 + d$ that uses factoring.

Have students complete Questions 1 and 2 with a partner or in groups. Share responses as a class.

Questions to ask

- · What operation or operations can you perform to isolate the variable?
- Which Property of Equality is associated with each operation?
- When do you need to take the square root before isolating the variable?
- How does the solution process of isolating a variable and then taking a square root compare to the solution process of taking a square root and then isolating the variable?
- How did setting one side of the equation equal to 0 factor into Oscar's solution strategy?
- Which property or properties did Oscar use in his strategy?

Summary

You can use Properties of Equality to solve quadratic equations in the form $y = x^2 + d$, $y = (x - c)^2$, $y = a(x - c)^2$, or $y = a(x - c)^2 + d$.

Activity 5.1 Solving Quadratic Equations by Factoring



DEVELOP

Facilitation Notes

In this activity, a Worked Example shows how to use factoring and the Zero Product Property to determine the roots of a given quadratic equation. Students then practice solving quadratic equations by factoring.

Ask a student to read the introduction aloud. Analyze the Worked Example and complete Question 1 as a class.

Differentiation strategies

- · To scaffold support, have students number the steps in the Worked Example.
- To extend the activity, discuss the relationship between the Worked Example and the roots in factored form, $y = a(x - r_1)(x - r_2)$.

Misconception

Students can use technology to identify the values of x that make the equation true by graphing the parabola $x^2 - 4x$ and the horizontal line y = -3. In this case, the solutions are the points of intersection between the two graphs. When solving the equation algebraically, students must set the quadratic expression equal to zero, factor the quadratic, and then use the Zero Product Property. In this case, they are actually determining the roots, or x-intercepts. The graphical representation of this solution is the intersection of the parabola and the horizontal line y = 0.

Have students complete Question 2 with a partner or in groups and share responses as a class.

Questions to ask

- How are the errors in each student's work similar?
- What strategy did you use to correctly solve the equation?
- · What do the solutions mean in terms of the given equations?

Have students work with a partner or in a group to complete Questions 3 and 4. Share responses as a class.

As students work, look for

- · Products that are correct, but do not lead to the correct sum for the middle term.
- · Sign errors.
- Problems factoring the polynomial in Question 3 part (c) because of the fractional coefficients. Remind students that they can multiply both sides of the equation by 6 to make the coefficients whole numbers.
- The realization that the $f(x) = x^2 + 10x + 12$ cannot be factored.

Questions to ask

- · What are the different ways the constant can be expressed as a product of two numbers?
- Is the middle term the sum of the two numbers? How do you know?
- Why does a factor (x c) lead to a root x = c?
- How can you check that your answer is correct?
- · How can you rewrite an equation without fractions?
- What do the solutions mean in terms of the given equation?
- Do you think an equation that cannot be factored can still have roots? Explain.

Differentiation strategy

To extend the activity, have students use technology to verify the solutions graphically.

Summary

One method to solve a quadratic trinomial of the form $ax^2 + bx + c = d$ is to use the Properties of Equality so that the quadratic expression equals zero, factor the quadratic expression, and then determine the roots using the Zero Product Property.

Activity 5.2Completing the Square to Determine Roots



Facilitation Notes

In this activity, students analyze a Worked Example that demonstrates how to complete the square to determine the zeros of the quadratic function that is unfactorable. Students then complete the square to determine the roots of other quadratic equations both in and out of context.

Ask a student to read the introduction aloud. Analyze the Worked Example as a class.

Questions to ask

- How is this Worked Example different than the work you did in the previous activity?
- Why is the algebraic expression set equal to zero?
- What is the first step in the Worked Example?
- Why is the *c*-value equal to 25?
- Why is it necessary to add 25 to the right side of the equation?
- Once the trinomial is rewritten as a perfect square trinomial, how are the steps familiar?
- What do the solutions mean in terms of the given equation?
- Revisit the graph of this function in the previous activity. How can you interpret the solutions in terms of the graph?

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

Questions to ask

- What value is substituted for *y*?
- What term is moved to the other side of the equation?
- What constant is added to both sides of the equation to complete the square?
- · What is the trinomial in factored form?
- How do you solve the remainder of the problem to get both roots?

Have students work with a partner or in a group to complete Questions 2 and 3. Share responses as a class.

Misconception

Students may think that because they used one method to start on the solution path that they cannot use a different method at any point. For example, when a quadratic equation is written in general form, the zeros may be easiest to identify using graphing technology. Encourage students to try different solution paths.

Questions to ask

- How can you use technology to determine the maximum height of the ball?
- What intervals can you use to show a complete graph?
- How would you describe the shape of the graph?
- How is the vertex related to the maximum point on the parabola?
- Is the height of the ball related to the *x* or *y*-coordinate of the maximum point?
- How could you solve this problem using the completing the square method?
- How did you label the two parallel sides of the rectangular plot that both have fencing?
- How did you label the side of the plot that is parallel to the house?
- What does your equation represent?
- What does the vertex of the parabola represent in this problem situation?
- What is another way to determine the dimensions that create the rectangular plot with the greatest area?

Summary

You can complete the square to determine the roots of a quadratic equation.

Activity 5.3

Using the Quadratic Formula



Facilitation Notes

In this activity, students review the Quadratic Formula as a method to calculate the roots of any quadratic equation written in general form. They analyze a Worked Example demonstrating how to use the Quadratic Formula to calculate the roots of a quadratic equation. They then use the formula to solve a problem in context. Students identify and correct a common error that occurs when using the Quadratic Formula and then use the Quadratic Formula to calculate the zeros of other functions.

Analyze the Worked Example as a class.

Differentiation strategies

To assist all students,

- Suggest that they use parentheses when substituting in values. This helps students avoid errors, particularly when the *b*-value is negative.
- Advise that they complete the problem in stages, rather than entering the entire expression in the calculator at one time.

Questions to ask

- Why must the quadratic equation appear in standard form to use the Quadratic Formula?
- Why do you have to separate your work into two equations?
- How can you check your solutions?
- What would you get if you substituted either x = -5.5 or x = -4.5 into the quadratic equation?
- How can you use a graph to check your solutions?

Have students work with a partner or in a group to read the scenario and complete Questions 1 and 2. Share responses as a class.

Questions to ask

- Draw a sketch to represent this problem situation.
- In which form is the quadratic function describing the launched T-shirts written?
- What are the values of a, b, and c?
- Do both roots make sense in the problem situation? Why or why not?

Differentiation strategies

To extend the activity,

- Have students solve this problem using graphing technology.
- · Have students research how to program the Quadratic Formula into a graphing calculator.

Have students work with a partner or in a group to complete Questions 3 and 4. Share responses as a class.

Questions to ask

- Did Meredith use the correct values for a, b, and c?
- · How is the form in which Meredith's equation is written different than the form of the equations you solved in the previous two questions?
- Is Meredith's equation set equal to 0?
- How can you rewrite Meredith's equation so it is equal to 0?
- What are the values for a, b, and c?
- What value appears under the radical after all operations are performed?
- What value appears in the denominator?

Summary

The Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, is a tool to calculate the roots to any quadratic equation written in standard form, $ax^2 + bx + c = 0$.

DEMONSTRATE

Talk the Talk: Show Me the Ways

Facilitation Notes

In this activity, students determine the roots of a quadratic equation using four methods: factoring, completing the square, using the Quadratic Formula, and graphing.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

As students work, look for

The same results using the four different methods. All methods should result in identifying the same two roots, x = -3 and x = 1.

Questions to ask

- Which method did you complete first? Why?
- Did all methods result in the same two roots?
- · Based on the form of this quadratic equation, which method do you prefer? Why?
- Did you use technology? If so, how?

Differentiation strategy

To extend the activity, discuss the four methods used to determine the real roots of the quadratic equation in Question 1. Have students explain circumstances in which each method might be preferable over the others. Then, have students create equations that would be most efficiently solved using each method.

Summary

Factoring, completing the square, using the Quadratic Formula, and graphing are all methods used to solve for the roots of quadratic equations.

NOTES

The Root of the Problem

Solving Quadratic Equations

Warm Up

Use the Distributive Property to determine each product.

1.
$$(x + 1)(x + 2)$$

2.
$$(x + 4)(x - 5)$$

3.
$$(2x - 3)(x - 4)$$

$$4.(x + 2)^2$$

Learning Goals

- · Factor quadratic trinomials to determine the roots of quadratic equations and to rewrite quadratic functions in forms that reveal different key characteristics.
- Complete the square to determine the roots of quadratic equations of the form $ax^2 + bx + c$.
- Use the Quadratic Formula to determine roots and zeros.

Key Term

· Quadratic Formula

You have analyzed the different structures of quadratic equations. How can the structure of a quadratic equation help you determine a solution strategy? Is there a single strategy that works to solve any quadratic equation?

LESSON 5: The Root of the Problem • 1

Warm Up Answers

- 1. $x^2 + 3x + 2$
- 2. $x^2 x 20$
- 3. $2x^2 11x + 12$
- $4. x^2 + 4x + 4$

- 1. Sample answers.
- 1a. $27 = x^2 9$

$$27 + 9 = x^2 - 9 + 9$$

Addition Property of Equality

$$\sqrt{36} = \sqrt{x^2}$$

Square Root Property

- $\pm 6 = x$
- 1b. $(x + 3)^2 = 121$

$$\sqrt{(x+3)^2} = \sqrt{121}$$

Square Root Property

$$x + 3 - 3 = -3 \pm 11$$

Subtraction Property of Equality

$$x = 8, -14$$

1c. $48 = 3(x - 1)^2$

$$\frac{48}{3} = \frac{3(x-1)^2}{3}$$

Division Property

 $\sqrt{16} = \sqrt{(x-1)^2}$ Square Root Property

 $1 \pm 4 = x - 1 + 1$ Addition Property

of Equality

$$x = 5, -3$$

1d.
$$\frac{1}{2}(x+5)^2 - 18 = 0$$

$$\frac{1}{2}(x+5)^2-18+18$$

$$= 0 + 18$$

Addition Property

of Equality

$$2 \cdot \frac{1}{2}(x+5)^2 = 2 \cdot 18$$

Multiplication Property of Equality

$$\sqrt{(x+5)^2} = \sqrt{36}$$

Square Root Property
$$x + 5 - 5 = -5 \pm 6$$

Subtraction Property

of Equality

$$x = 1, -11$$

GETTING STARTED

Grassroots

You know how to use the Properties of Equality to solve equations in the forms shown.

$$y = x^{2} + d$$

$$y = (x - c)^{2}$$

$$y = a(x - c)^{2}$$

- $y = a(x c)^2 + d$
- 1. Use Properties of Equality to solve each equation. State the property used in each step of your solution.

a.
$$27 = x^2 - 9$$

b.
$$(x + 3)^2 = 121$$

c.
$$48 = 3(x - 1)^2$$

d.
$$\frac{1}{2}(x+5)^2-18=0$$

2. Describe the strategy Oscar used to solve part (a) in Question 1.

$$27 = x^2 - 9$$

$$27 - 27 = x^2 - 9 - 27$$

$$0 = x^2 - 36$$

$$0 = (x - 6)(x + 6)$$

$$(x - 6) = 0$$
 and $(x + 6) = 0$
 $x = 6$ and $x = -6$

- 2 · TOPIC 2: Exploring and Analyzing Patterns
- 2. Oscar used the Subtraction Property of Equality to set the quadratic expression equal to 0 and then factored the quadratic expression. He then set each factor equal to 0 to determine the solutions.

ACTIVITY 5.1

Solving Quadratic Equations by Factoring



Let's consider strategies to solve quadratics in the form $y = ax^2 + bx + c$ using the factoring strategies you have learned.

Worked Example

You can use factoring to calculate the roots for the quadratic equation $x^2 - 4x = -3$.

$$x^{2} - 4x = -3$$

$$x^{2} - 4x + 3 = -3 + 3$$

$$x^{2} - 4x + 3 = 0$$

$$(x - 3)(x - 1) = 0$$

$$(x-3) = 0$$
 and $(x-1) = 0$
 $x-3+3=0+3$ and $x-1+1=0+1$
 $x=3$ and $x=1$

1. Why is 3 added to both sides in the first step of the **Worked Example?**



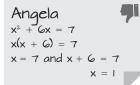
The Zero Product Property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

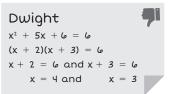


What is the connection between the Worked Example and determining the roots from factored form, $y = a(x - r_1)(x - r_2)$?



2. Determine each student's error and then solve each equation correctly.





LESSON 5: The Root of the Problem · 3

Answers

- 1. To keep both sides of the equation balanced, you must perform the same operation on both sides of the equals sign.
- 2. Sample answer. Neither student set the equation equal to zero before factoring. You can only use the Zero Product Property if the product of the factors is equal to zero.

Angela's equation

$$x^2 + 6x - 7 = 0$$

$$(x + 7)(x - 1) = 0$$

$$x = -7 \text{ and } x = 1$$

Dwight's equation

$$x^2 + 5x = 0$$

$$x(x+5)=0$$

$$x = 0$$
 and $x = -5$

3a. x = 6 and x = 2

3b.
$$x = -7$$
 and $x = -1$

3c.
$$x = 0$$
 and $x = \frac{5}{4}$

3d. cannot be factored

4. Jim used the Properties of Equality because the quadratic equation only had one variable term. Pam factored the quadratic equation using the difference of two squares and then used the Zero Product Property to determine the two solutions.

3. Use factoring to solve each quadratic equation, if possible.

a.
$$x^2 - 8x + 12 = 0$$

b.
$$x^2 + 8x = -7$$



What efficiency strategies did you use to solve linear equations with fractional coefficients?

c.
$$\frac{2}{3}x^2 - \frac{5}{6}x = 0$$

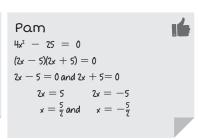
d.
$$f(x) = x^2 + 10x + 12$$

4. Describe the different strategies and reasoning that Jim and Pam used to solve $4x^2 - 25 = 0$.

Jim

$$4x^2 - 25 = 0$$

 $4x^2 = 25$
 $x^2 = \frac{25}{4}$
 $x = \pm \sqrt{\frac{25}{4}}$
 $x = \pm \frac{5}{2}$



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5.2

Completing the Square to Determine Roots



You can use the completing the square method to determine the roots of a quadratic equation that cannot be factored.

Worked Example

Complete the square to determine the roots of the equation $x^2 + 10x + 12 = 0$.

Isolate
$$x^2 + 10x$$
. $x^2 + 10x + 12 - 12 = 0 - 12$
 $x^2 + 10x = -12$

Determine the constant term that would complete the square. Add this term to both sides of the equation. $x^2 + 10x + 25 = -12 + 25$ $x^2 + 10x + 25 = 13$

Rewrite the left side as a $(x + 5)^2 = 13$ perfect square.

Take the square root of each $\sqrt{(x+5)^2} = \pm \sqrt{13}$ side of the equation. $x+5=\pm \sqrt{13}$

The roots are approximately -1.39 and -8.61.



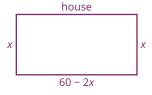
How was equality of the equation maintained through the completing the square process?

LESSON 5: The Root of the Problem • 5

1a.
$$x = 3 \pm \sqrt{5}$$

1b.
$$x = 6 \pm \sqrt{30}$$

- 2. The maximum height of the ball is 20 feet.
- 3. The dimensions of the rectangular plot with the maximum area are 15 feet for the width and 30 feet for the length parallel to the house. The maximum area is 450 square feet.



1. Complete the square to determine the roots of each equation.

a.
$$x^2 - 6x + 4 = 0$$

b.
$$x^2 - 12x + 6 = 0$$

- 2. A ball is thrown straight up from 4 feet above the ground with a velocity of 32 feet per second. The height of the ball over time can be modeled with the function $h(t) = -16t^2 + 32t + 4$. What is the maximum height of the ball?
- 3. Jessie is fencing in a rectangular plot outside of her back door so that she can let her dogs out to play. She has 60 feet of fencing and only needs to place it on three sides of the rectangular plot because the fourth side will be bound by her house. What dimensions should Jesse use for the plot so that the maximum area is enclosed? What is the maximum area? Draw a diagram to support your work.

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ACTIVITY 5.3

Using the Quadratic Formula



The **Quadratic Formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c represent real numbers and $a \neq 0$.

Worked Example

You can use the Quadratic Formula to determine the zeros of the function $f(x) = -4x^2 - 40x - 99$.

Rewrite the function as an

equation to be solved for $x -4x^2 - 40x - 99 = 0$

when y = 0.

Determine the values of

a, b, and *c*.

a = -4, b = -40, c = -99

Substitute the values into the Quadratic Formula.

rewrite the expression.

Perform operations to

 $X = \frac{-(-40) \pm \sqrt{(-40)^2 - 4(-4)(-99)}}{2}$

 $x = \frac{40 \pm \sqrt{1600 - 1584}}{2}$

 $x = \frac{40+4}{-8}$ and $x = \frac{40-4}{-8}$

and $x = \frac{36}{-8}$

x = -5.5 and x = -4.5

The zeros of the function $f(x) = -4x^2 - 40x - 99$ are x = -5.5 and x = -4.5.

The Seaside Serpents baseball team has a new promotional activity to encourage fans to attend games: launching free T-shirts! They can launch a T-shirt in the air with an initial velocity of 91 feet per second from $5\frac{1}{2}$ feet off the ground (the height of the team mascot).

A T-shirt's height can be modeled with the quadratic function $h(t) = -16t^2 + 91t + 5.5$, where t is the time in seconds and h(t) is the height of the launched T-shirt in feet. They want to know how long it will take for a T-shirt to land back on the ground after being launched (if no fans grab it before then!).



What would a sketch showing the height of the T-shirt over time look like?

LESSON 5: The Root of the Problem • 7

1. It makes sense to use the Quadratic Formula because the quadratic equation is not factorable and completing the square would be difficult with the values of the coefficients.

2.

$$x = \frac{-91 \pm \sqrt{91^2 - 4(-16)(5.5)}}{2(-16)}$$

$$= \frac{-91 \pm \sqrt{8633}}{-32}$$

$$x \approx -0.0598 \text{ and}$$

$$x \approx 5.747$$

The T-shirt will take about 5.75 seconds to land back on the ground after being launched.

3a. Meredith determined the roots of the equation $y = x^2 - 7x - 8$, not of the equation $y = x^2 - 7x - 11$. To determine the roots of a quadratic equation using the Quadratic Formula, the equation must be set equal to 0.

3b.
$$x = \frac{7 \pm \sqrt{93}}{2}$$

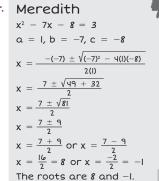
 $x \approx 8.32$ and $x \approx -1.32$
4a. $x = -5.10$ and
 $x = -0.10$
4b. $x \approx -3.84$ and $x \approx 0.17$

1. Why does it make sense to use the Quadratic Formula to solve this problem?



Do you think an exact solution or approximate solution is more appropriate for this context?

- Use the Quadratic Formula to determine how long it will take for a T-shirt to land back on the ground after being launched.
- 3. Meredith is solving the quadratic equation $x^2 7x 8 = 3$. Her work is shown.
 - a. Identify Meredith's error.

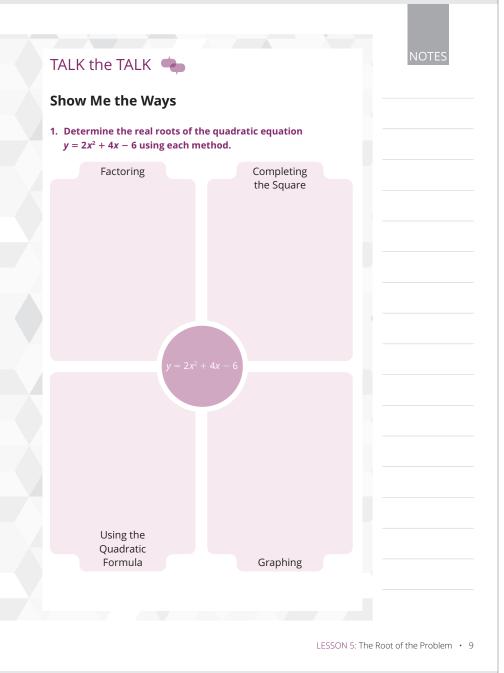


- b. Determine the solution to Meredith's quadratic equation.
- 4. Use the Quadratic Formula to determine the zeros for each function. Round the solutions to the nearest hundredth.

a.
$$f(x) = 2x^2 + 10x - 1.02$$

b.
$$h(x) = 3x^2 + 11x - 2$$

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1.
$$y = 2x^2 + 4x - 6$$

 $2x^2 + 4x - 6 = 0$

Factoring:

$$2(x+3)(x-1) = 0$$

$$x = -3, x = 1$$

Completing the Square:

$$2x^2 + 4x = 6$$

$$2(x^2+2x)=6$$

$$x^2 + 2x = 3$$

$$x^2 + 2x + 1 = 4$$

$$(x + 1)^2 = 4$$

$$x = -1 \pm 2$$

$$x = -3, x = 1$$

Quadratic Formula:

$$a = 2, b = 4, c = -6$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-6)}}{2(2)}$$

$$\chi = \frac{-4 \pm \sqrt{64}}{4}$$

$$X = \frac{-4 \pm 8}{4}$$

$$x = -3, x = 1$$

Graphing:

