Exploring and Analyzing Patterns Summary

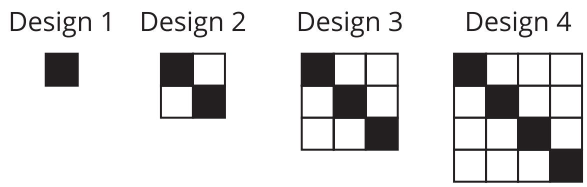
KEY TERMS

- relation
- function
- function notation
- standard form of a quadratic function
- factored form of a quadratic function
- vertex form of a quadratic function
- concavity of a parabola
- Quadratic Formula
- the number *i*
- imaginary roots
- imaginary zeros
- complex numbers
- real part of a complex number
- imaginary part of a complex number
- imaginary numbers
- pure imaginary number
- Fundamental Theorem of Algebra

LESSON 1 Patterns: They're Grrrrrowing!

Many patterns can be described mathematically using diagrams, algebraic expressions or functions, and words . Sequences can be used to show observable patterns. To analyze a pattern, you can draw a picture, sketch a graph, create a table, or write an equation. Analyzing a pattern can help you to recognize the pattern and extend it.

For example, consider the tile pattern shown.



The total number of tiles in each design can be described by the sequence {1, 4, 9, 16, . . .}. Each term in the sequence is a square of the term number.

LESSON 2

The Cat's Out of the Bag!

A visual model, a table of values, and a graph can be used to identify patterns as linear, exponential, or quadratic.

For example, in the previous tile pattern, a table of values comparing the number of black tiles and white tiles in each design can be created to determine the number of white tiles in Design n.

Design Number	Total Number of Tiles	Number of Black Tiles	Number of White Tiles
1	1	1	0
2	4	2	2
3	9	3	6
4	16	4	12
n	n^2	n	$n^2 - n$

The pattern is quadratic.

Two or more algebraic expressions are equivalent if they produce the same output for all input values. You can verify that two expressions are equivalent by using properties to rewrite the two expressions as the same expression.

For example, Lincoln says that the number of white tiles in each row of the design is one less than the design number and expresses this pattern with the expression $n(n - 1)$. The expression $n(n - 1) = n^2 - n$ since both produce the same output for all input values.

LESSON 3

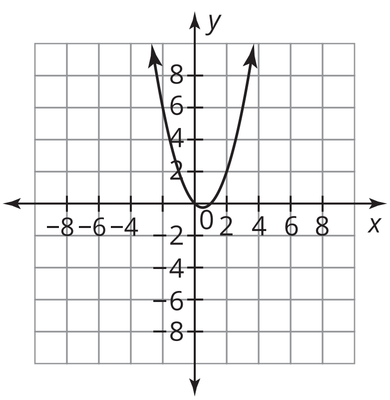
Samesies

Relationships between quantities can be represented in graphs, tables, equations, and contexts.

A **relation** is a mapping between a set of input values and a set of output values. A **function** is a relation such that for each element of the domain there exists exactly one element in the range. In **function notation**, the function $f(x)$ is read as "f of x" and indicates that x is the input and $f(x)$ is the output.

For example, in the previous tile pattern, the number of white tiles, $f(x)$, can be represented by the function $f(x) = x^2 - x$, where x represents the design number.

It can also be represented by the graph and the table shown.



x	y
1	1
2	2
3	6
4	12

Two functions are equivalent if their algebraic or graphical representations are the same.

LESSON 4

True to Form

Quadratic functions can be written in different forms.

Standard form: $f(x) = ax^2 + bx + c$, where a does not equal 0.
Factored form: $f(x) = a(x - r_1)(x - r_2)$, where a does not equal 0.
Vertex form: $f(x) = a(x - h)^2 + k$, where a does not equal 0.

The graphs of quadratic functions can be described using key characteristics: x-intercept(s), y-intercept, vertex, axis of symmetry, and concavity. The key characteristics of a function can be determined using different methods depending on the form of the function.

Standard Form Methods to Determine Key Characteristics	
Axis of Symmetry $x = \frac{-b}{2a}$	***x*-intercept(s)** Substitute 0 for y, and then solve for x using the Quadratic Formula, factoring, or graphing technology.
Vertex Use $\frac{-b}{2a}$ to determine the x-coordinate of the vertex. Then substitute that value into the equation and solve for y.	***y*-intercept** c-value

Factored Form Methods to Determine Key Characteristics	
Axis of Symmetry $x = \frac{r_1 + r_2}{2}$	***x*-intercept(s)** $(r_1, 0)$, $(r_2, 0)$
Vertex Use $\frac{r_1 + r_2}{2}$ to determine the x-coordinate of the vertex. Then substitute that value into the equation and solve for y.	***y*-intercept** Substitute 0 for x, and then solve for y.

Vertex Form Methods to Determine Key Characteristics	
Axis of Symmetry $x = h$	***x*-intercept(s)** Substitute 0 for y, and then solve for x using the Quadratic Formula, factoring, or graphing technology.
Vertex (h, k)	***y*-intercept** Substitute 0 for x, and then solve for y.

Concavity of a parabola describes whether a parabola opens up or opens down. A parabola is concave up if it opens upward; a parabola is concave down if it opens downward. When the leading coefficient a is negative, the graph of the quadratic function opens downward and has a maximum. When a is positive, the graph of the quadratic function opens upward and has a minimum.

You can write a unique quadratic function given a vertex and a point on the parabola.

For example, consider a parabola with vertex (5, 2) that passes through the point (4, 9).

Substitute the given values into the vertex form of the function and solve for a.

$$f(x) = a(x - h)^2 + k$$
$$9 = a(4 - 5)^2 + 2$$
$$9 = a(-1)^2 + 2$$
$$9 = 1a + 2$$
$$7 = a$$

Finally, substitute the a-value into the function. $f(x) = 7(x - 5)^2 + 2$

You can write a unique quadratic function given the roots and a point on the parabola.

For example, consider a parabola with roots at (−2, 0) and (4, 0) that passes through the point (1, 6).

Substitute the given values into the vertex form of the function and solve for a.

$$f(x) = a(x - r_1)(x - r_2)$$
$$6 = a(1 + 2)(1 - 4)$$
$$6 = a(3)(-3)$$
$$6 = -9a$$
$$-\frac{2}{3} = a$$

Finally, substitute the a-value into the function. $f(x) = -\frac{2}{3}(x + 2)(x - 4)$

You can determine a unique quadratic function algebraically given three reference points. Substitute the x- and y-values of each point into the standard form, $y = ax^2 + bx + c$ to create a system of three equations. Then, use elimination and substitution to solve for a, b, and c.

LESSON 5

The Root of the Problem

You can use factoring and the Zero Product Property to solve quadratics in the form $y = ax^2 + bx + c$.

For example, you can solve the quadratic equation $x^2 - 4x = -3$.

$$x^2 - 4x = -3$$
$$x^2 - 4x + 3 = -3 + 3$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$

$$(x - 3) = 0 \quad \text{or} \quad (x - 1) = 0$$
$$x - 3 + 3 = 0 + 3 \quad \text{or} \quad x - 1 + 1 = 0 + 1$$
$$x = 3 \quad \text{or} \quad x = 1$$

For a quadratic function that has zeros but cannot be factored, there is another method for solving the quadratic equation. **Completing the square** is a process for writing a quadratic expression in vertex form, which then allows you to solve for the zeros.

For example, you can calculate the roots of the equation $x^2 - 4x + 2 = 0$.

Isolate $x^2 - 4x$.

$$x^2 - 4x + 2 - 2 = 0 - 2$$
$$x^2 - 4x = -2$$

Determine the constant term that would complete the square.
Add this term to both sides of the equation.

$$x^2 - 4x + ? = -2 + ?$$
$$x^2 - 4x + 4 = -2 + 4$$
$$x^2 - 4x + 4 = 2$$

Factor the left side of the equation.
Determine the square root of each side of the equation.

$$(x - 2)^2 = 2$$
$$\sqrt{(x - 2)^2} = \sqrt{2}$$
$$(x - 2) = \pm\sqrt{2}$$

Set the factor of the perfect square trinomial equal to each square root of the constant and solve for x.

$$x - 2 = \sqrt{2} \quad \text{or} \quad x - 2 = -\sqrt{2}$$
$$x = 2 + \sqrt{2} \quad \text{or} \quad x = 2 - \sqrt{2}$$
$$x \approx 3.41 \quad \text{or} \quad x \approx 0.59$$

The roots are approximately 3.41 and 0.59.

The **Quadratic Formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b and c represent real numbers and $a \neq 0$.

For example, given the function $f(x) = 2x^2 - 4x - 3$ you can identify the values of a, b and c.

$$a = 2;\ b = -4;\ c = -3$$

Then you use the quadratic formula to solve.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{16 + 24}}{4}$$

$$x = \frac{4 \pm \sqrt{40}}{4}$$

$$x \approx \frac{4 + 6.325}{4} \approx 2.581 \quad \text{or} \quad x \approx \frac{4 - 6.325}{4} \approx -0.581$$

The roots are $x \approx 2.581$ and $x \approx -0.581$.

You can use the same methods you used to solve a system of two linear equations to solve a system of two quadratic equations.

For example, consider the system shown.

$$\begin{cases} y = x^2 + 3x - 5 \\ y = -x^2 + 10x - 1 \end{cases}$$

Use substitution to set the two equations equal to each other.

$$x^2 + 3x - 5 = -x^2 + 10x - 1$$

Solve for x.

$$2x^2 - 7x - 4 = 0$$
$$(2x + 1)(x - 4) = 0$$
$$x = -\frac{1}{2} \text{ or } x = 4$$

Substitute each x-value into one of the original equations to solve for y.

$$y = \left(-\frac{1}{2}\right)^2 + 3\left(-\frac{1}{2}\right) - 5 \qquad y = (4)^2 + 3(4) - 5$$
$$= \frac{1}{4} - \frac{3}{2} - 5 \qquad = 16 + 12 - 5$$
$$= -6\frac{1}{4} \qquad = 23$$

The solutions to the system are $(-\frac{1}{2}, -6\frac{1}{4})$ and $(4, 23)$.

LESSON 6

i Want to Believe

In order to calculate the square root of any real number, there must be some way to calculate the square root of a negative number. That is, there must be a number such that when it is squared, it is equal to a negative number. For this reason, mathematicians defined what is called **the number *i***. The number i is a number such that $i^2 = -1$.

For example, you can rewrite the expression $\sqrt{-25}$ using i.

Factor out -1.	$\sqrt{-25} = \sqrt{(-1)(25)}$
Rewrite the radical expression.	$= \sqrt{-1} \cdot \sqrt{25}$
Apply the square root on $\sqrt{25}$.	$= 5\sqrt{-1}$
Rewrite $\sqrt{-1}$ as i.	$= 5i$

So, $\sqrt{-25}$ can be rewritten as $5i$.

Functions and equations that have imaginary solutions have **imaginary roots** or **imaginary zeros**, which are the solutions.

The set of **complex numbers** is the set of all numbers written in the form $a + bi$, where a and b are real numbers. The term a is called the **real part of a complex number**, and the term bi is called the **imaginary part of a complex number**.

The set of **imaginary numbers** is a subset of the set of complex numbers. A **pure imaginary number** is a number of the form $a + bi$, where b is not equal to 0.

When operating with complex numbers involving i, combine like terms by treating i as a variable (even thought it is a constant).

For example, consider the sum of $(2 + 3i) + (-5 + i)$.

$$\begin{aligned}(2 + 3i) + (-5 + i) &= (2 + (-5)) + (3i + \mathrm{i}) \\ &= -3 + 4i\end{aligned}$$

You can also multiply complex numbers using the Distributive Property.

For example, consider the product of $(2 + i)(2 - i)$.

$$\begin{aligned}(2 + i)(2 - i) &= 2(2) + 2(-i) + i(2) + i(-i) \\ &= 4 - 2i + 2i - i^2 \\ &= 4 - i^2 \\ &= 4 - (-1) = 5\end{aligned}$$

You can use the same methods you used to solve quadratic equations with real solutions to solve quadratic equations with imaginary solutions.

For example, consider the function $f(x) = x^2 - 2x + 2$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 - 8}}{2}$$

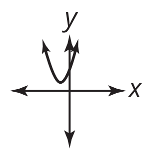
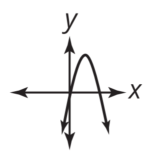
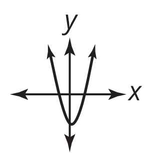
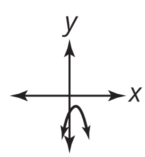
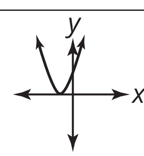
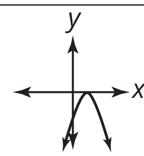
$$x = \frac{2 \pm \sqrt{-4}}{2}$$

$$x = \frac{2 \pm 2i}{2}$$

$$x = 1 \pm i$$

The **Fundamental Theorem of Algebra** states that any polynomial equation of degree *n* must have, exactly *n* complex roots or solutions.

Any root may be a multiple root. The table shows the number of real and imaginary roots for different quadratic equations.

Location of Vertex	Concavity	Sketch	Number of *x*-Intercepts	Number and Type of Roots
Above the *x*-axis	Up		0	2 imaginary roots
	Down		2	2 real roots
Below the *x*-axis	Up		2	2 real roots
	Down		0	2 imaginary roots
On the *x*-axis	Up		1	1 unique root
	Down		1	1 unique root