1

Ahead of the Curve

Solving Quadratic Inequalities

MATERIALS

Graphing technology

Lesson Overview

Students analyze a Worked Example to calculate the solution set of a quadratic inequality by first solving for the roots of its related quadratic equation, then determining which interval(s) created by the roots satisfy the inequality. They use both a number line and coordinate plane to select the correct intervals and then make connections between those methods. Students solve a problem in context requiring the use of a quadratic inequality, and also use a transformation to make comparisons within a context. Throughout this lesson, students use the Quadratic Formula, technology, and inequality or interval notation.

Algebra 2

Quadratic and Square Root Functions, Equations, and Inequalities

(4) The student applies mathematical processes to understand that quadratic and square root functions, equations, and quadratic inequalities can be used to model situations, solve problems, and make predictions. The student is expected to:

(H) solve quadratic inequalities.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 5.B, 5.F, 5.G

Essential Ideas

- A horizontal line drawn across the graph of a quadratic function intersects the parabola at exactly two points, except at the vertex, where it intersects the parabola at exactly one point.
- The solution set of a quadratic inequality is determined by first solving for the roots of the quadratic equation, and then determining which interval(s) created by the roots will satisfy the inequality. A combination of algebraic and graphical methods may be the most efficient solution method.
- Quadratic inequalities can be used to model some real-world contexts. The effects of translations of quadratic functions can be used to make comparisons within a context.

Lesson Structure and Pacing: 1 Day

Engage

Getting Started: It Has Its Ups and Downs

Students use a graph that models a vertical motion problem to approximate the times when an object is at specified heights. They recognize that a horizontal line drawn on the graph always intersects it at two points, except at the vertex. Students also identify regions on the graph that are less than or greater than a given height and write a quadratic inequality to represent their responses.

Develop

Activity 1.1: Solving Quadratic Inequalities

Students analyze a Worked Example that demonstrates how to solve a quadratic inequality. First, they rewrite the inequality as an equation and calculate the roots. Students then divide a number line into intervals based upon the roots. Next, they test an x-value within each interval to determine whether values that lie in the interval satisfy the inequality. Finally, students write an inequality statement based upon their results.

Activity 1.2: Modeling Quadratic Inequalities

Students define the variables and write a function to represent a vertical motion problem. They use technology to graph the function and then determine when the object is above or below specific heights. Students combine their algebraic solution for the roots and visual inspection of the graph to determine the correct intervals. The context of the problem must also be considered when determining appropriate intervals for the solution.

Demonstrate

Talk the Talk: Boom! Boom!

Students analyze the effect of a translation of a quadratic function within a context.

Facilitation Notes

In this activity, students use a graph that models a vertical motion problem to approximate the times when an object is at specified heights. They recognize that a horizontal line drawn on the graph always intersects it at two points, except at the vertex. Students also identify regions on the graph that are less than or greater than a given height and write a quadratic inequality to represent their responses.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

As students work, look for

- Responses that include only one time for when the firework is at 1000 ft, 2500 ft, and 3900 ft. If students cannot see two responses for 3900 feet, have them use technology to zoom into that portion of the graph. Discuss the usefulness of knowing that the vertex, or maximum point, is (15.625, 3911.25). Therefore, there must be two times when the firework is at 3900 feet.
- Two responses for times when the firework is at 0 feet. While the parabola intersects the *x*-axis in two places, only one of the responses makes sense in the context.
- The response $0 \le t < 4.7$ and $26.55 < t \le 31.26$ to Question 3, part (c). While these inequalities reflect the correct times, the question asks for a quadratic inequality, which would be $-16t^2 + 500t + 5 < 2000$.

Misconception

Students may think the parabola is the actual path of the firework as opposed to a relationship between time in seconds and height in feet. Remind students that the path is not parabolic when an object is shot straight into the air and falls to the ground.

Questions to ask

- Why is there only one time when the firework is 0 feet off the ground?
- Why are the two times very close to each other when the firework is 3900 feet off the ground?
- What is the average for the two responses for each question? What is the significance of that value in relation to the graph?
- What is the maximum height the firework reaches?
- What is the total number of seconds the firework is above 2000 feet?

- What is the total number of seconds the firework is below 2000 feet?
- Did you use an equal sign with each of your inequalities? Why or why not?
- Write an inequality for when the firework is above 2000 feet.

Summary

A horizontal line drawn across the graph of a quadratic function intersects the parabola at exactly two points, except at the vertex, where it intersects the parabola at exactly one point.

DEVELOP

Activity 1.1

Solving Quadratic Inequalities

Facilitation Notes

In this activity, students analyze a Worked Example that demonstrates how to solve a quadratic inequality. First, they rewrite the inequality as an equation and calculate the roots. Students then divide a number line into intervals based upon the roots. Next, they test an *x*-value within each interval to determine whether values that lie in the interval satisfy the inequality. Finally, students write an inequality statement based upon their results.

Analyze the Worked Example as a class.

Questions to ask

- How was the quadratic inequality written as a quadratic equation?
- How were the roots of the quadratic equation determined?
- Explain how the number line was set up.
- How do you know what values to substitute into the inequality?
- How was the solution interval determined?

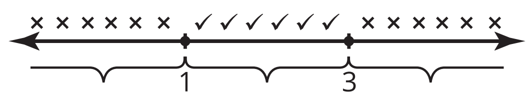
Misconception

Students may confuse the interval notation (1, 3) with the coordinate pair (1, 3). Discuss the importance of using the entire notation $x \in (1, 3)$ so it is understood that both values in the parentheses relate to x.

Differentiation strategies

- To assist all students, allow them to use inequality notation or interval notation. Include additional directions for using interval notation that will be needed for Question 3.

   - A combination of parentheses and brackets are acceptable. For example, $x \in [1, 3)$ means the same as $1 \le x < 3$.
   - Parentheses are used with the infinity symbol. For example, $x \in (-\infty, 3)$ means the same as $x < 3$.
- To extend the lesson, have students use factored form $(x - 3)(x - 1) = 0$ to test values from each interval. While some students may favor the concrete values in the Worked Example, others may find it more efficient to avoid calculations with numbers and just deal with signs. Demonstrate how to use factored form, then have students try both methods as they analyze Jeff's work in Question 3. Discuss students' preferences and reasoning.



Interval 1	Interval 2	Interval 3
Try $x = 0$	Try $x = 2$	Try $x = 4$
$(x - 3)(x - 1)$	$(x - 3)(x - 1)$	$(x - 3)(x - 1)$
$(-)(-) = +$	$(-)(+) = -$	$(+)(+) = +$

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Questions to ask

- What five points did you use to sketch the parabola?
- What key characteristics are obvious in the quadratic equation $y = x^2 - 4x + 3$?
- What key characteristics are obvious when the quadratic equation is written in factored form?
- How did you determine the vertex of $y = x^2 - 4x + 3$?
- What point is symmetric to the y-intercept?
- Which axis on the graph relates to the number line used in the Worked Example?
- How many intervals does the x-axis divide the parabola into?
- How can you tell from the graph which intervals satisfy the inequality?
- How does this graph represent what is shown on the number line in the Worked Example?
- What method do you prefer to solve a quadratic inequality: Using the number line and testing numbers in each interval, or graphing the parabola and using the graph to determine which intervals satisfy the inequality? Why?

Have students work with a partner or in a group to analyze the student work and complete Question 3. Share responses as a class.

Differentiation strategies

- To assist all students, suggest they use a number line as they analyze Jeff's work.
- To extend the lesson, have students use factored form $2(x - 5)(x - 2) = 0$ to test values from each interval.

Questions to ask

- What is the difference between the roots of a quadratic equation and the solution set of a quadratic inequality?
- Why is it helpful to know the roots when determining the solution to a quadratic inequality?
- After determining the roots of a quadratic equation, how is the solution set to the associated quadratic inequality determined?
- Why did Jeff choose the values $x = 1$, $x = 3$, and $x = 6$? Where did these values come from?
- What would happen if Jeff used the roots rather than values within each interval to substitute into the inequality?
- Should the values $x = 1$, $x = 3$, and $x = 6$ be used to describe the solution? Explain.
- Did Jeff base his solution set on the values he chose in each interval or on the roots of the quadratic equation?
- What advice would you give Jeff so that he does not make the same mistake again?

Summary

The solution set of a quadratic inequality is determined by first solving for the roots of the associated quadratic equation, then determining which interval(s) determined by the roots satisfy the inequality.

Activity 1.2

Modeling Quadratic Inequalities

Facilitation Notes

In this activity, students define the variables and write a function to represent a vertical motion problem. They use technology to graph the function and then determine when the object is above or below specific heights. Students combine their algebraic solution for the roots and visual inspection of the graph to determine the correct intervals. The context of the problem must also be considered when determining appropriate intervals for the solution.

Differentiation strategy

To extend the lesson, prior to engaging in the questions, ask students to create questions that can be answered using this scenario.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

Questions to ask

- Why does it make sense that a parabola represents vertical motion?
- How did you know what coefficients to use for the variables?
- What does each term of the trinomial mean with respect to the problem situation?
- Can this quadratic equation be easily factored?
- What methods could be used to solve this quadratic equation?

Have students work with a partner or in a group to complete Questions 2 through 4. Share responses as a class.

Misconception

Students may be confused as to why they are solving for the roots when the heights are 30 and 43. If this is the case, remind them that algebraically they set the quadratic equation equal to zero; when the equation is set equal to zero, they are identifying the roots. If helpful, provide an example of a geometric transformation that demonstrates their algebraic process; the x-values of the points of intersection of the horizontal line and parabola are equivalent to the roots of the function when its graph is translated vertically.

As students work, look for

- Exclusion of the endpoints in Question 3, part (b) and inclusion of the endpoints in Question 4.
- A solution for Question 4 that takes into account the context. Rather than a solution for the function, $t \in (-\infty, 1.375]$ or $t \in [1.5, \infty)$, the solution should be $t \in [0, 1.375]$ or $t \in [1.5, 3.1]$, which relates to the context, where the initial time is 0 seconds, and the balloon lands on the ground at about 3.1 seconds.
- Use of the Quadratic Formula or technology to determine the times when the balloon is at or below 43 feet.

Questions to ask

- As the time increases, does the height increase or decrease?
- What method did you use to solve the inequality?
- Why did you use the Quadratic Formula to solve the inequality?
- Is your solution an exact solution or an approximate solution? Why?
- How is having a graph of the situation available helpful in determining the intervals when the balloon is above 30 feet?
- Why was it unnecessary to create a number line and test intervals?

- Did you use the same method to determine when the height was less than or equal to 43 feet? If not, what method did you use?
- What inequality sign is associated with the inequality representing the times when the balloon is at or below 43 feet?
- How do you know where each interval on the graph begins and ends?
- How many intervals is the graph divided into?
- Which interval(s) satisfy the inequality?

Summary

Quadratic inequalities can be used to model some real-world contexts. A combination of algebraic and graphical solutions may be the most efficient solution method.

DEMONSTRATE

Talk the Talk: Boom! Boom!

Facilitation Notes

In this activity, students analyze the effect of a translation of a quadratic function within a context.

Have students work with a partner or in a group to complete Questions 1 through 5. Share responses as a class.

As students work, look for

- The realization that the second firework is an example of a vertical translation.
- Awareness that the question is asking about the *x*-value, time, while a vertical translation affects the *y*-values.
- Different predictions, and the magnitude of the change predicted.

Misconception

Because students understand that the vertical translation affects the height, they may be confused as to why the *x*-value, time, is affected.

- Use the graph of the function to help them make sense of the situation. If the function is lowered on the coordinate plane, the span of *x*-values above the horizontal line $y = 2000$ is more narrow. This means the interval of time for the translated function is less than the interval of time for the original function.
- Algebraically, if the *y*-value is held constant at 2000 for both functions, the *x*-values that correspond to 2000 will be different for each function.

Differentiation strategies

To scaffold support,

- Suggest that students access the tables for the functions to make comparisons.
- Provide viewing windows so that students can see the difference between the two graphs at meaningful points.

	Launch	Maximum	Landing
Xmin	0	15	31
Xmax	0.1	16	31.5
Ymin	0	3900	0
Ymax	50	3915	50

Questions to ask

- What was your prediction based on?
- Do you think the second firework will reach the same height as the first firework? Explain.
- Do you think the second firework will land on the ground at the same time as the first firework? Explain.
- If the firework is launched from the ground, what is the value of the constant, h_0, in the vertical motion equation $h(t) = -16t^2 + v_0t + h_0$?
- Is the function for the second firework an example of a horizontal or vertical translation of the original function?
- How was 2000 used to write the inequality?
- Did you use the Quadratic Formula or technology to determine the intervals for when the firework will be above 2000 feet? Why did you choose that method?
- What do the values $x = 4.7$ and $x = 26.5$ represent?

Summary

The effects of translations of quadratic functions can be used to make comparisons within a context.

NOTES

Warm Up Answers

1. $x = \pm 6$
2. $x = -5$ and $x = 2$
3. $x = \frac{1}{2}$ and $x = -\frac{7}{2}$
4. $x = -2 \pm 2\sqrt{3}$

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Ahead of the Curve

Solving Quadratic Inequalities

Warm Up

Determine the solution of each quadratic equation.

1. $x^2 - 100 = -64$

2. $x^2 + 3x + 5 = 15$

3. $4x^2 + 12x = 7$

4. $x^2 + 4x - 3 = 5$

Learning Goals

- Solve a quadratic inequality by calculating the roots of the quadratic equation which corresponds to the inequality and testing values within intervals determined by the roots.
- Connect the graphical representation of a quadratic function and the solution to a corresponding quadratic inequality represented on a number line.
- Use interval notation to record the solutions to quadratic inequalities.

You have interpreted the solution sets to linear inequalities on a coordinate plane. You have also solved quadratic equations using a variety of methods. How can you interpret the solutions sets to quadratic inequalities on a coordinate plane using what you know about solving quadratic equations?

LESSON 1: Ahead of the Curve • 1

Answers

Check student work for approximations.

1a. About −0.01 seconds and 31.26 seconds

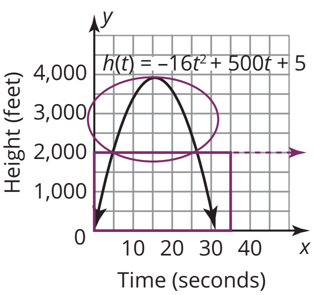
1b. About 2.14 seconds and 29.11 seconds

1c. About 6.23 seconds and 25.02 seconds

1d. About 14.79 seconds and 16.46 seconds

2. Sample answer.
For most heights, the firework will reach the height at two different times, as it travels upward and then downward. It will reach the maximum height at one time only. Because the firework has an initial height of 5 feet, it will reach heights less than 5 feet only once, as it travels downward.

3a. and 3b.



3c. $-16t^2 + 500t + 5 < 2000$

GETTING STARTED

It Has Its Ups and Downs

A firework is shot straight up into the air with an initial velocity of 500 feet per second from 5 feet off the ground. The graph of the function that represents this situation is shown.

Remember:

A vertical motion model is a quadratic equation of the form $y = -16t^2 + v_0t + h_0$.

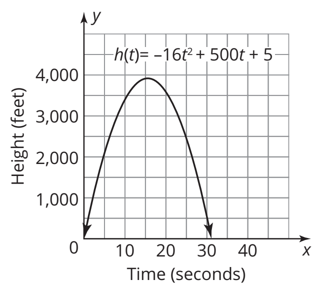
1. Use the graph to approximate when the firework will be at each given height off the ground.

a. 0 feet **b. 1000 feet**

c. 2500 feet **d. 3900 feet**

2. Describe any patterns you notice for the number of times the firework reaches a given height.

3. Draw a horizontal line on the graph to represent when the firework is 2000 feet off the ground.



a. When is the firework higher than 2000 feet? Circle this portion of the graph.

b. When is the firework below 2000 feet? Draw a box around this portion of the graph.

c. Write a quadratic inequality that represents the times when the firework is below 2000 feet.

ELL Tip

One term that is used throughout the activity is *initial velocity*. Define *initial* as *the beginning or first occurrence of something*, or *the original amount*. Define *velocity* as *the speed of an object in a specific direction*. Read aloud the sentences at the beginning of the section. Discuss the term *initial velocity* in the context of the sentences as *the beginning speed* of the firework that was shot up in the air. Then discuss why the *velocity* changes, when it increases, and when it decreases.

Another term that is used throughout the activity is *portion*. Define *portion* as *a section or part of something*. In this activity, students are asked to identify *portions*, or sections, of a graph.

ACTIVITY
1.1 Solving Quadratic Inequalities

Just like with the other inequalities you have studied, the solution to a quadratic inequality is the set of values that satisfy the inequality.

Worked Example

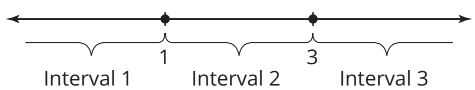
Let's determine the solution of the quadratic inequality $x^2 - 4x + 3 < 0$.

Write the corresponding quadratic equation. $x^2 - 4x + 3 = 0$

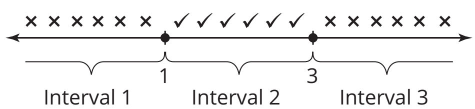
Calculate the roots of the quadratic equation using an appropriate method.

$$(x - 3)(x - 1) = 0$$
$$(x - 3) = 0 \quad \text{or} \quad (x - 1) = 0$$
$$x = 3 \quad \text{or} \quad x = 1$$

Plot the roots to divide the number line into three regions.



Choose a value from each interval to test in the original inequality.



Try $x = 0$	Try $x = 2$	Try $x = 4$
$0^2 - 4(0) + 3 < 0$	$2^2 - 4(2) + 3 < 0$	$4^2 - 4(4) + 3 < 0$
$3 < 0$ ✗	$4 - 8 + 3 < 0$	$16 - 16 + 3 < 0$
	$-1 < 0$ ✓	$3 < 0$ ✗

Identify the solution set as the interval(s) in which your test value satisfies the inequality.

Interval 2 satisfies the original inequality, so the solution includes all numbers between 1 and 3.

Solution: $x \in (1, 3)$

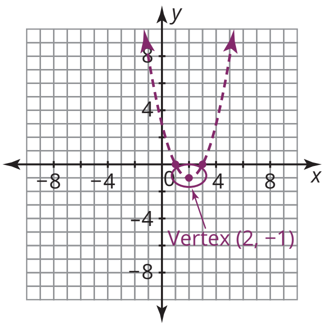
The symbol $\in$ is read "is an element of," "is in," or "belongs to." The notation $x \in (1, 3)$ has the same meaning as $1 < x < 3$.

Answers

1a. The solution would include the endpoints, $x \in [1, 3]$.

1b. Solution: $x \in (-\infty, 1]$ or $x \in [3, \infty)$
The solution would include all numbers less than or equal to 1 and greater than or equal to 3.

2.



Sample answer.
The x-axis on the graph represents the number line shown in the Worked Example. The interval that I circled represents the solution because these are the x-values that are between 1 and 3. The parts of the graph that are not circled represent the intervals that are not solutions because they are less than or equal to 1 and greater than or equal to 3.

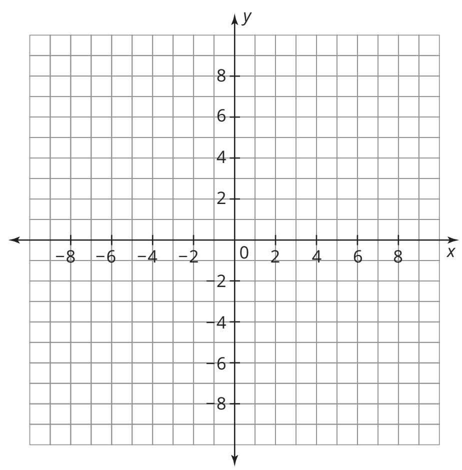
The notation $x \in [1, 3]$ means the same as $1 \le x \le 3$.

1. Analyze the Worked Example.

a. How would the solution set change if the inequality was less than or equal to? Explain your reasoning.

b. How would the solution set change if the inequality was greater than or equal to? Explain your reasoning.

2. Graph $y = x^2 - 4x + 3$ on the coordinate plane shown and label the roots of the equation and the vertex. Then describe how the graph supports that the solution set for the associated quadratic inequality $x^2 - 4x + 3 < 0$ is $1 < x < 3$.



Jeff correctly determined the roots of the quadratic inequality $2x^2 - 14x + 27 \geq 7$ to be $x = 5$ and $x = 2$. However, he incorrectly determined the solution set. His work is shown.

Jeff

$$2x^2 - 14x + 27 = 7$$
$$2x^2 - 14x + 20 = 0$$
$$2(x^2 - 7x + 10) = 0$$
$$2(x - 5)(x - 2) = 0$$
$$x = 5 \quad \text{or} \quad x = 2$$

$x = 1$	$x = 3$	$x = 6$
$2(1)^2 - 14(1) + 27 \geq 7$	$2(3)^2 - 14(3) + 27 \geq 7$	$2(6)^2 - 14(6) + 27 \geq 7$
$2 - 14 + 27 \geq 7$	$18 - 42 + 27 \geq 7$	$72 - 84 + 27 \geq 7$
$15 \geq 7$ ✓	$3 \geq 7$ ✗	$15 \geq 7$ ✓

Solution: $x \in (-\infty, 1]$ or $x \in [6, \infty)$

3. Describe Jeff's error. Then, determine the correct solution set for the inequality.

Ask yourself:

When testing values from each interval, could you use the factored form of the inequality rather than the original inequality?

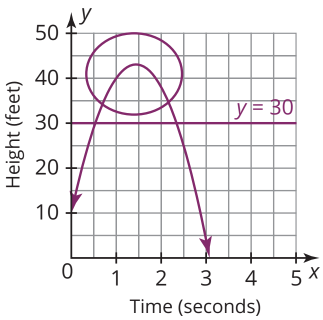
Answer

3. Jeff based his solution set on his chosen values instead of on the roots of the quadratic equation. The correct solution set is $x \in (-\infty, 2]$ or $x \in [5, \infty)$.

Answers

1. Let t represent the time in seconds, and let h represent the height of the balloon in feet t seconds after it is thrown.
$h(t) = -16t^2 + 46t + 10$

2.



3a. See graph.
$t \in (0.53, 2.34)$

3b. $-16t^2 + 46t + 10 > 30$

$-16t^2 + 46t + 10 = 30$

$-16t^2 + 46t - 20 = 0$

$a = -16, b = 46, c = -20$

$t = \frac{-46 \pm \sqrt{836}}{-32}$

$t \approx 0.534$ or $t \approx 2.341$

Solution: $t \in (0.534, 2.341)$

The balloon is above 30 feet between 0.534 and 2.341 seconds.
I can tell this is the solution because the interval that I circled on the graph is between those roots.

4. $-16t^2 + 46t + 10 \leq 43$

$-16t^2 + 46t + 10 = 43$

$-16t^2 + 46t - 33 = 0$

$a = -16, b = 46, c = -33$

$t = \frac{-46 \pm 2}{-32}$

$t = 1.375$ or $t = 1.5$

Solution: $t \in [0, 1.375]$ or $t \in [1.5, 3.1]$

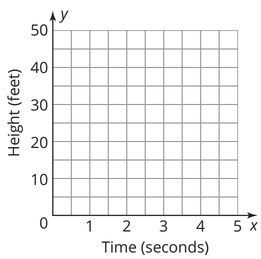
ACTIVITY **1.2**

Modeling Quadratic Inequalities

A water balloon is launched from a machine upward from a height of 10 feet with an initial velocity of 46 feet per second.

1. Identify the variables and write a quadratic function to represent this situation.

2. Use technology to sketch the graph of the function.



3. Draw a horizontal line on the graph to represent when the balloon is 30 feet off the ground.

a. Circle the portion of the graph that represents when the balloon is above 30 feet.

b. Write and solve an inequality to determine when the balloon is above 30 feet. Use the graph to explain your solution.

4. Determine when the balloon is at or below 43 feet. Interpret your solution in terms of the model you graphed.

The balloon is less than 43 feet between 0 and 1.375 seconds, because time in this scenario cannot be negative. The balloon is also less than 43 feet between 1.5 seconds and 3.1 seconds, since the balloon reaches the ground at about 3.1 seconds.

TALK the TALK

Boom! Boom!

In the Getting Started activity, a firework was shot straight up into the air with an initial velocity of 500 feet per second from 5 feet off the ground. The function representing the situation was identified as $h(t) = -16t^2 + 500t + 5$. You determined the firework would be above 2000 feet between about 5 seconds and 27 seconds.

Suppose a second firework was shot straight up into the air with an initial velocity of 500 feet per second from the ground.

1. **Predict whether the second firework will be above 2000 feet for more time, less time, or the same amount of time as the first firework.**

2. **Write a quadratic inequality to represent when the second firework will be above 2000 feet.**

3. **Determine when the second firework will be above 2000 feet.**

NOTES

Answers

1. Sample answer.
   I predict the second firework will be above 2000 feet for less time than the first firework because it starts from a lower height.
2. $-16t^2 + 500t > 2000$
3. $-16t^2 + 500t > 2000$

   $-16t^2 + 500t = 2000$

   $-16t^2 + 500t - 2000 = 0$

   $a = -16, b = 500,$
   $c = -2000$

   $t = \frac{-125 \pm 5\sqrt{305}}{-8}$

   $t \approx 4.7$ or $t \approx 26.5$

   The second firework will be above 2000 feet for $t \in (4.7, 26.5)$.

Answers

4. Sample answer.
   Yes, the second firework is in the air for less time, but by only 0.2 seconds.
5. Sample answers.
   The graph of the second firework appears to be the same graph as the first firework because of the scale.
   The graph of the second firework is translated 5 units down from the graph of the first firework.

NOTES

4. Was your prediction made in Question 1 correct?

5. Use technology to compare the graph of the first firework to the graph of the second firework. What do you notice?