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The Ol' Switcharoo

Inverses of Linear and Quadratic Functions

MATERIALS

Patty paper
Graphing Technology

Lesson Overview

Students are introduced to the *inverse of a function*. A Worked Example highlights how to determine the inverse of a linear function algebraically. Students use this example to determine other inverses of functions. They then create the graph of the inverse of a linear function by reflecting the original function across the line $y = x$ using patty paper. This process is repeated for quadratic functions. The term *one-to-one function* is defined, and students determine whether the inverse of a function is also a function. A graphic organizer is completed to summarize the definition and representations of inverses functions.

Algebra 2
Attributes of Functions and their Inverses

(2) The student applies mathematical processes to understand that functions have distinct key attributes and understand the relationship between a function and its inverse. The student is expected to:

(B) graph and write the inverse of a function using notation such as $f^{-1}(x)$.

(C) describe and analyze the relationship between a function and its inverse (quadratic and square root, logarithmic and exponential), including the restriction(s) on domain, which will restrict its range.

ELPS

1.A, 1.D, 1.E, 1.G, 2.C, 2.D, 2.G, 2.H, 2.I, 3.A, 3.B, 3.C, 3.D, 3.F, 4.A, 4.B, 4.C, 4.G, 4.K, 5.B, 5.E

Essential Ideas

- Inverses of functions can be determined algebraically and graphically.
- The inverse of a function is determined by replacing $f(x)$ with y, switching the x and y variables, and solving for y.
- The graph of the inverse of a function is a reflection of that function across the line $y = x$.
- To restrict the domain of a function means to define a new domain for the function that is a subset of the original domain.
- A one-to-one function is a function in which its inverse is also a function.
- For a one-to-one function $f(x)$, the notation for its inverse is $f^{-1}(x)$.

Lesson Structure and Pacing: 3 Days

Day 1

Engage

Getting Started: Inside an Enigma

Students write a coded note using a numeric rule that they devise. They then decode their classmates' notes and discuss how the inputs and outputs of the rules they used to decode the notes compare with those used to encode the notes. Students also discuss why it is important that the cipher rules they created are functions—that they assign each element of the domain to exactly one element of the range.

Develop

Activity 3.1: The Inverse of a Function

Students are presented with data about the exchange rate between lira and U.S. dollars. They complete a table and write an equation to represent the relationship to convert dollars to lira. They then use their intuitive understanding of inverses to complete a table and write an equation for the inverse situation, to convert lira to dollars. The *inverse of a function* is then formally defined and a Worked Example demonstrates how to determine the inverse of a function algebraically.

Activity 3.2: Graphing Inverses of Functions

Students begin by graphing the original function from the previous activity along with the line $y = x$ on patty paper. They reflect $f(x)$ across the line $y = x$ to observe that the inverse of a function is a reflection of the original function across this line. Students generalize that given a point (a, b) on the original function $f(x)$, there is a corresponding point (b, a) on the graph of its inverse.

Day 2

Activity 3.3: Exploring Inverses of Quadratics

Students use patty paper to sketch the graph of the inverse of a quadratic function. A Worked Example shows how to algebraically determine the inverse. Students reason that the inverse of a quadratic function is not a function because the square root of any number other than 0 has a positive and negative value. The term *restrict the domain* is defined. Students analyze a graph of the quadratic function and the positive and negative square root. They identify the restricted domain of the original function to produce each equation of the inverse. Students see that the domain and range of the inverse of a function is the reverse of the domain and range of an original function.

Activity 3.4: More With Inverses of Quadratics

Students deal with a context involving vertical motion. They restrict the domain based on the problem situation and graph the function with that restricted domain. Students then graph the inverse of the function with the restricted domain. This exercise causes students to focus on why domains are sometimes restricted, to have reasonable inputs for a given context and guarantee that an original equation and its inverse can both be functions. After algebraically determining the inverse of the function, students use the inverse to answer questions related to the problem situation.

Day 3

Activity 3.5: One-to-One Functions

Students learn that a *one-to-one function* is a function whose inverse is also a function. Students create tables of values, plot points, and graph given functions and their inverses, then determine whether the original function is a one-to-one function. They conclude that all linear functions, except for constant functions, have inverses that are functions. A graphic organizer is used to describe strategies to identify and determine inverses of functions.

Demonstrate

Talk the Talk: 1-2-1

Students analyze a situation to conclude that not all linear functions are one-to-one functions. They conclude that linear functions are sometimes one-to-one functions, exponential functions are always one-to-one functions, and linear absolute value functions and quadratic functions are never one-to-one functions.

Getting Started: Inside an Enigma

ENGAGE

Facilitation Notes

In this activity, students write a coded note using a numeric rule that they devise. They then decode their classmates' notes and discuss how the inputs and outputs of the rules they used to decode the notes compare with those used to encode the notes. Students also discuss why it is important that the cipher rules they created are functions—that they assign each element of the domain to exactly one element of the range.

Ask a student to read the introduction aloud. Discuss the information as a class.

Have students work with a partner or in a group to complete Questions 1 through 4. Share responses as a class.

Misconception

Students may assume the decoding rules are limited to using one operation because of the example. Mathematical rules are not limited to one operation. They can be written using multiple operations such as $2x + 3$; it will just be more time consuming to decode the rule that was applied.

Questions to ask

- What substitution cipher rule did you use?
- What operation(s) did you use in your substitution cipher rule?
- How many operations can you use in a substitution cipher rule?
- Is your substitution cipher rule a function?
- What would happen if an element of the domain could be assigned to more than one element of the range?

Summary

A function can take the outputs of another function as its inputs.

DEVELOP

Activity 3.1

The Inverse of a Function

Facilitation Notes

In this activity, students are presented with data about the exchange rate between lira and U.S. dollars. They complete a table and write an equation to represent the relationship to convert dollars to lira. They then use their intuitive understanding of inverses to complete a table and write an equation for the inverse situation, to convert lira to dollars. The *inverse of a function* is then formally defined, and a Worked Example demonstrates how to determine the inverse of a function algebraically.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Differentiation strategies

- To scaffold support, suggest that students think of a Turkish lira as they would a quarter. The relationship between quarters and dollars is the same as the relationship between lira and dollars: 4:1.
- To assist all students, suggest they use variables other than x and y to avoid confusion as to which variable is the input and which variable is the output when dealing with inverses.

Questions to ask

- What operation is used to convert U.S. currency into Turkish currency?
- Is 100 dollars equal to 25 lira or is 100 dollars equal to 400 lira?
- What variables did you use to write the equation? What does each variable represent?
- Does the equation represent a function?

Have students work with a partner or in a group to complete Questions 3 through 6. Share responses as a class.

Misconceptions

- Students may be confused by use of the term *inverse*. In the past it has been associated with the term *multiplicative inverse* and *additive inverse*. Given the expression x, the multiplicative inverse is $\frac{1}{x}$ and the additive inverse is $-x$. Ask the students to explain the difference between the inverse of a function, the multiplicative inverse, and the additive inverse.
- Students may incorrectly connect an inverse relationship to a reciprocal relationship due to the equations used in this activity, $f(x) = 4x$ and $f^{-1}(x) = \frac{1}{4}x$. This misunderstanding can be dispelled by providing the students with an example of a different function and its inverse, e.g., $f(x) = x + 8$.

Questions to ask

- What operation is used to convert Turkish currency into U.S. currency?
- Is 400 lira equal to 100 dollars or 1600 dollars?
- Which column in the table represents the input? How do you know?
- Did the input and output columns switch places?
- Was the input in the initial equation rewritten as the output in the inverse equation?
- Did you switch the variables when you wrote the inverse equation?
- What variables did you use to write the inverse equation? What does each variable represent?
- Do you think that the inverse of a function is always a function?

Ask a student to read the information and definition aloud following Question 6. Analyze the Worked Example as a class.

Have student work with a partner or in a group to complete Question 7. Share responses as a class.

Questions to ask

- Why does it make sense to interchange the input and output values when writing the inverse of a function?
- Why do you have to solve the equation for the output value?

Summary

The outputs of a function are the inputs of its inverse. The inputs of a function are the outputs of its inverse.

Activity 3.2
Graphing Inverses of Functions

Facilitation Notes

In this activity, students begin by graphing the original function from the previous activity along with the line $y = x$ on patty paper. They reflect $f(x)$ across the line $y = x$ to observe that the inverse of a function is a reflection of the original function across this line. Students generalize that given a point (a, b) on the original function $f(x)$, there is a corresponding point (b, a) on the graph of its inverse.

Begin by distributing a piece of patty paper to each student. Have students work with a partner or in a group to read the Worked Example and complete Questions 1 through 3. Share responses as a class.

Differentiation strategy

To assist all students, do the patty paper activity as a class and do it *without* first graphing the inverse of the function. Let the students see the image that resulted from the reflection across the line $y = x$ before they actually graph the inverse. Then when they graph the inverse of the original function, they will notice it is exactly the same line as the reflection image.

As students work, look for

Different methods to place the coordinate plane on the patty paper.

- Students may display the first quadrant or all four quadrants.
- Students may draw or fold to get the axes.

Misconception

Students may not make the connection between the algebraic method to determine an inverse and the graphic method to determine an inverse. When studying transformations, students typically used axes or other horizontal or vertical lines of reflection. In this situation, they use a diagonal line of reflection. Ask students why the line $y = x$ is used as the line of reflection rather than the x- or y-axis. What is the algebraic method? What is the connection between the reflection line $y = x$ and the algebraic method?

Questions to ask

- Where are the x- and y-axes on your patty paper?
- What points did you use to graph the function $f(x) = 4x$ on your patty paper?
- What points did you use to graph the function $f(x) = \frac{1}{4}x$ on your patty paper?
- What points did you use to graph the line $y = x$ on your patty paper?
- Does $y = \frac{1}{4}x$ describe the line $y = 4x$ reflected across the line $y = x$?

- What is the slope of the original equation? What is the slope of the inverse equation?
- How is the slope of the original equation related to the slope of the inverse equation?
- Why is there a reciprocal relationship between the slope of the original equation and the slope of the inverse equation?

Summary

The graph of the inverse of a function is a reflection of that function across the line $y = x$.

Activity 3.3

Exploring Inverses of Quadratics

Facilitation Notes

In this activity, students use patty paper to sketch the graph of the inverse of a quadratic function. A Worked Example shows how to algebraically determine the inverse. Students reason that the inverse of a quadratic function is not a function because the square root of every number other than 0 has a positive and negative value. The term *restrict the domain* is defined. Students analyze a graph of the quadratic function and the positive and negative square root. They identify the restricted domain of the original function to produce each equation of the inverse. Students see that the domain and range of the inverse of a function is the reverse of the domain and range of an original function.

Provide patty paper to students to complete Question 1.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

As students work, look for

Missing connections related to the line of reflection $y = x$. Students may not understand why this particular line is the line of reflection rather than one of the axes or any other line on the graph. Practice relating original points and the points to which they are mapped to help students make this connection. Connect this understanding to the worked example where students algebraically determine the inverse function by switching the x and y variables.

Questions to ask

- When the parabola is reflected across the line $y = x$, what point does the point (2, 4) map to? Why does this happen?
- When graphing the inverse for any function, is the line of reflection always the line $y = x$? Why is that the case?
- What is the definition of a function?
- How does the Vertical Line Test relate to the definition of a function?
- Does the graph of the inverse of the quadratic function pass the Vertical Line Test?
- What is the relationship between the domain and range of the quadratic function and the domain and range of the inverse of the function?

Analyze the Worked Example. Discuss and complete Question 4 as a class.

Questions to ask

- What are the steps for algebraically determining the inverse of a linear function?
- How are the steps for algebraically determining the inverse of a quadratic function different than the steps for determining the inverse of a linear function?
- Why is the ± required?
- How does the ± relate to the graph of the inverse of the quadratic function?
- Does the square root of a number always have two values? Explain.
- Does each x-value have a unique y-value?
- Why did you have to use two equations to enter the equation containing the ± symbol?

Ask a student to read the definition aloud and discuss as a class. Have students work with a partner or in a group to complete Question 5. Share responses as a class.

Questions to ask

- How did you use the graph to write the restricted domain of $f(x) = x^2$?
- Why was it important to restrict the domain?

Differentiation strategy

To extend the lesson, give students a transformed quadratic function. Ask them to determine the domain and range, identify the inverse of the function, and then determine the domain and range of the inverse of the function. For example, consider $y = x^2 + 3$, where the domain is all real numbers and the range is $y \geq 3$. The inverse of the function is $y = \pm\sqrt{x - 3}$, where the domain is $x \geq 3$ and the range is all real numbers.

Summary

Algebraically determining the inverse of a quadratic function is the same process as determining the inverse of a linear function; however, both the positive and negative roots must be considered. You can restrict the domain of a function to create an inverse that is also a function.

Activity 3.4

More with Inverses of Quadratics

Facilitation Notes

In this activity, students deal with a context involving vertical motion. They restrict the domain based on the problem situation and graph the function with that restricted domain. Students then graph the inverse of the function with the restricted domain. This exercise allows students to focus on why domains are sometimes restricted, to have reasonable inputs for a given context and guarantee that an original equation and its inverse can both be functions. After algebraically determining the inverse of the function, students use the inverse to answer questions related to the problem situation.

Ask a student to read the introduction aloud. Discuss as a class.

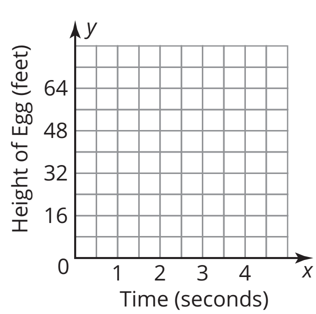
Questions to ask

- How tall is the building? How do you know?
- Why is there no *x*-term? What does this mean with respect to the problem situation?
- What is the shape of the graph of the function?
- What key characteristics can be identified using the equation?
- Does the graph of the equation have a minimum or maximum point? How do you know?

Have students work with a partner or in a group to complete Questions 1 through 4. Share responses as a class.

Differentiation strategies

- To scaffold support, provide a scale for the graph in Question 4.



- To assist all students, take the opportunity to examine the graph in Question 4. Use this graph as an example of why sometimes you cannot see the symmetry of the graph of a quadratic function.

Questions to ask

- Does the time depend on the height of the egg, or does the height of the egg depend on the time?
- Why is the range of the function less than or equal to 64?
- How is the graph of the problem situation different than the graph of the equation?

Have students work with a partner or in a group to complete Questions 5 through 7. Share responses as a class.

Questions to ask

- How did you determine the domain and range of the inverse of the function?
- How did you graph the inverse of the function? Did you use a table of values or the graph above?
- Does the graph of the inverse of $f(x)$ look like the graph of $f(x)$ reflected across the line $y = x$?

Have students work with a partner or in a group to complete Questions 8 through 11. Share responses as a class.

Questions to ask

- If $x = -16y^2 + 64$, what does y equal?
- Which function, $f(x)$ or its inverse, models the height of the egg based on the time the egg is in the air? Models the time the egg is in the air based on the height of the egg?
- If you are given the time the egg is in the air and want to determine the height of the egg, is it easier to use $f(x)$ or its inverse? Why?
- If you are given the height of the egg and want to determine the time the egg is in the air, is it easier to use $f(x)$ or its inverse? Why?
- What equation did you use to determine the height of the egg after 1.5 seconds? Why?
- What equation did you use to determine the time when the egg is 55 feet in the air? Why?

Summary

When a problem requires using a given function to determine the independent quantity when a dependent quantity is given, determining the inverse of the original function may be a more efficient way to handle the situation.

Activity 3.5
One-to-One Functions

Facilitation Notes

In this activity, students learn that a *one-to-one function* is a function whose inverse is also a function. Students create tables of values, plot points, and graph given functions and their inverses, then determine whether the original function is a one-to-one function. They conclude that all linear functions, except for constant functions, have inverses that are functions. A graphic organizer is used to describe strategies to identify and determine inverses of functions.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

Differentiation strategy

To assist all students, provide patty paper to graph the function, and graph the line $y = x$. Then have them reflect the function to determine the graph of the inverse of the function.

As students work, look for

Use of the Vertical Line Test and the Horizontal Line Test. Remind students that a graph must pass the Vertical Line Test to be considered a function, and mention that if the graph passes the Horizontal Line Test, it is a one-to-one function.

Misconception

Students may not understand why the inverse of a function is only a function when the original function is a one-to-one function. If the original function is not one-to-one, then the inverse would be mapping y onto two or more x's. A good example of a function that is not one-to-one is any constant function. The inverse of any constant function is a vertical line which is not a function.

Questions to ask

- How did you determine the output for each function?
- How did you use the table of values for each function to complete the table of values for its inverse?
- How did you use the equation of the function to write the equation of its inverse?
- Does the graph of each function pass the Vertical Line Test?
- If the graph passes the Vertical Line Test, what does this imply?
- Does the graph of each function pass the Horizontal Line Test?
- If the graph passes the Horizontal Line Test, what does this imply?

Have students work with a partner or in a group to complete Questions 2 and 3. Share responses as a class.

Questions to ask

- Is $y = 6$ considered a linear function?
- How would you describe the graph of $y = 6$?
- What is the inverse equation of $y = 6$?
- How would you describe the graph of $x = 6$?
- Is $x = 6$ a function?
- How would you describe the linear functions that do have an inverse?
- Do all functions have inverses? Why or why not?
- Will any function that when graphed is symmetrical over $y = x$ have an inverse that is the same function?
- What is the inverse of the function $f(x) = x$?

Have students work with a partner or in a group to complete Question 4 Share responses as a class.

Questions to ask

- Which description involves switching the x and y variables in the equation before solving for y?
- Explain the steps to determine the inverse of a function algebraically.
- Which description involves reflecting $f(x)$ over the line $y = x$?
- Explain why this reflection method works.
- Which description involves switching the independent and dependent variables?
- Explain how switching the variables uses the definition of the inverse of a function.

Summary

Inverses can be described using a table of values, a verbal description, an equation, and a graph. When a function and its inverse are both functions, the original function is called a one-to-one function.

Talk the Talk: 1-2-1

DEMONSTRATE

Facilitation Notes

In this activity, students analyze a situation to conclude that not all linear functions are one-to-one functions. They conclude that linear functions are sometimes one-to-one functions, exponential functions are always one-to-one functions, and linear absolute value functions and quadratic functions are never one-to-one functions.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Questions to ask

- What special case is Adam overlooking?
- Do all lines pass the Vertical Line Test?
- How would you describe a line that fails the Vertical Line Test?
- What is the equation of a line that has an inverse which is not a function?
- What is the shape of an exponential function?
- Do all exponential functions and their inverses pass the Vertical Line Test?
- Do any quadratic functions and their inverses pass the Vertical Line Test?
- What is the shape of an absolute value function?
- Do any absolute value functions and their inverses pass the Vertical Line Test?
- Why might it be helpful to use both a Horizontal and Vertical Line Test to determine whether a function is one-to-one?

Summary

A function is a one-to-one function if both the function and its inverse are functions.

NOTES

Warm Up Answers

1. $x = 51$
2. $x = 720$
3. $x = 4$

3

The Ol' Switcharoo

Inverses of Linear and Quadratic Functions

Warm Up

Solve each equation.

1. $2x - 5 = 97$
2. $\frac{1}{3}x + 40 = 280$
3. $-4x - 10 = -26$

Learning Goals

- Determine the inverse of a given situation using words.
- Determine the inverse of a function numerically using a table, an equation, and a graph.
- Determine whether given functions are one-to-one functions.
- Identify function types that are always, sometimes, or never one-to-one functions.

Key Terms

- inverse of a function
- one-to-one function
- restrict the domain

You know that a function takes a set of inputs and maps them to a set of outputs. What happens when the outputs and inputs are reversed?

Answers

1. Answers will vary.
2. Answers will vary.
3. The cipher rule should be a function to ensure that there is no confusion in decoding. If an element of the domain (the letters of the alphabet) can be assigned to more than one element of the range (numeric values), then the code would be ambiguous.
4. The inputs of the decoding rule are the outputs of encoding rule, and the outputs of the decoding rule are the inputs of the encoding rule. The functions for encoding and decoding show inverse operations.

GETTING STARTED

Inside an Enigma

One of the simplest methods of creating a code is called a substitution cipher. For a substitution cipher, you can take each letter of the alphabet in numeric order and assign it to a different number using a mathematical rule.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

For example, if the cipher were written as $x + 4$, then L, which is currently assigned to 12, would be assigned to $12 + 4 = 16$, or P, in the code. The letter Y, which is currently assigned to 25, would be assigned to 3, or C, in the code. The word "inverse" would have the code "mrzivwi."

Cipher: $x + 4$

Word	Code
inverse	mrzivwi

1. **Write a substitution cipher rule. Then write a short note to a classmate in code. Give your rule and coded note to a classmate to decode.**

2. **Use mathematical notation to write the rule you used to decode your classmate's note.**

3. **Why is it important that the substitution cipher rule be a function?**

4. **Compare the inputs and outputs of the cipher rule you created and the rule used to decode your note. What do you notice?**

ELL Tip

Define the term *enigma* as *something that is mysterious or difficult to understand*. Provide examples of things that may be considered an *enigma*, such as *a scientific theory, a high-level crossword puzzle,* or a *confusing riddle*. Once students have read through the activity, encourage a discussion about why the title of the activity, "Inside an Enigma," is an appropriate title based on the definition of *enigma*. Define the term *cipher* as *a secret way of writing or a code*. Examples may include *computer codes, cell phone codes,* and *Morse code*. Discuss the *substitution cipher* used in the activity and how it relates to the definition of *cipher*.

ACTIVITY 3.1

The Inverse of a Function

Miguel is planning a trip to Turkey. Before he leaves, he wants to exchange his money to the Turkish lira, the official currency of Turkey. The exchange rate at the time of his trip is 4 lira per 1 U.S. dollar.

1. Complete the table of values to show the currency conversion for U.S. dollars to Turkish lira.

U.S. Currency (dollars)	Turkish Currency (lira)
100	
250	
400	
650	
1000	

2. Write an equation to represent the number of lira in terms of the number of U.S. dollars.

Suppose at the end of his trip, Miguel needs to convert any remaining lira to dollars. This situation is the *inverse* of the original situation.

3. What are the independent and dependent quantities of the inverse of the problem situation? How do these quantities compare to the quantities in Question 1?

4. Complete the table of values to show the inverse of the problem situation.

5. Compare the tables in Questions 1 and 4. What do you notice?

6. Use the table to write an equation for the inverse of the problem situation. Does this equation represent a function? Explain your answer.

ELL Tip

Clarify the meanings of the terms *currency* and *exchange rate*. Define *currency* as *a system of money for a country*, and *exchange rate* as *the value of one currency for conversion to another currency*.

Answers

1.

U.S. Currency (dollars)	Turkish Currency (lira)
100	400
250	1000
400	1600
650	2600
1000	4000

2. $y = 4x$

3. For the inverse situation, the dependent quantity is the amount in dollars, and the independent quantity is the amount in lira. These are the reverse of the quantities in Question 1.

4.

Turkish Currency (lira)	U.S. Currency (dollars)
400	100
1000	250
1600	400
2600	650
4000	1000

5. The columns are reversed.

6. $y = \frac{1}{4}x$; this equation does represent a function.

Answer

7. $f(x) = 4x$

$y = 4x$

$x = 4y$

$y = \frac{1}{4}x$

The inverse is the same equation that I wrote in Question 6.

Recall that a function takes an input value, performs some operation(s) on this value, and creates an output value. The **inverse of a function** takes the output value, performs some operation(s) on this value, and arrives back at the original function's input value. In other words, an inverse of a function "undoes" the function.

Worked Example

Given a function, $f(x)$, you can determine the inverse algebraically by following these steps.

Step 1: Replace the function $f(x)$ with another variable, generally y.

Step 2: Switch the x and y variables in the equation.

Step 3: Solve for y.

7. Use function notation to represent the number of lira $f(x)$ in terms of the number of U.S. dollars, x. Then complete the steps shown in the Worked Example to represent the number of U.S. dollars in terms of the number of lira. Compare the inverse to the equation you wrote in Question 6. What do you notice?

ACTIVITY
3.2

Graphing Inverses of Functions

In the previous activity you wrote the inverse of a linear function using algebra. Let's consider how to show the inverse of a function using its graph.

Worked Example

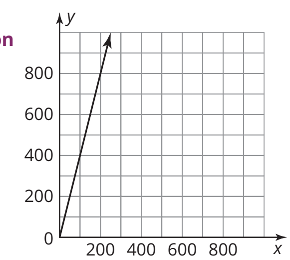
Given a function, $f(x)$, you can determine the inverse of a function graphically by following these steps.

Step 1: Copy the coordinate plane and graph $f(x)$ and the line $y = x$ onto patty paper.

Step 2: Heavily trace the graph of $f(x)$ with a pencil.

Step 3: Reflect the patty paper across the line $y = x$, and rub the paper so that the image of the graph of its inverse appears.

1. Consider the graph of the function $f(x) = 4x$ from the previous activity. Complete the steps in the Worked Example to graph the inverse using patty paper.



Think about:

How do the algebraic process and the graphical process to determine an inverse compare?

2. Compare the image you created and the graph of the inverse.

a. What do you notice about the image and the graph of the inverse?

b. What does this tell you about the graph of a function and its inverse and about the line $y = x$?

Answers

1. Check students' work.

2a. They are the same.

2b. The graph of the inverse of a function is a reflection of the function across the line $y = x$.
The line $y = x$ is the line of reflection of the function and the inverse of the function.

Answers

3a. The corresponding point on the graph of the inverse of $g(x)$ is (2, 3).

3b. The corresponding point on the graph of the inverse of $h(x)$ is (0, −1).

3c. The corresponding point on the graph of the inverse of $f(x)$ is (b, a).

NOTES

3. For each function and a given point on the graph of the function, determine the corresponding point on the graph of the inverse of the function.

a. Given that (3, 2) is a point on the graph of $g(x)$, what is the corresponding point on the graph of the inverse of $g(x)$?

b. Given that (−1, 0) is a point on the graph of $h(x)$, what is the corresponding point on the graph of the inverse of $h(x)$?

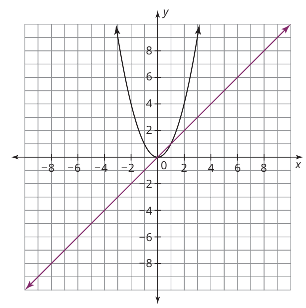
c. Given that (a, b) is a point on the graph of $f(x)$, what is the corresponding point on the graph of the inverse of $f(x)$?

ACTIVITY

3.3 Exploring Inverses of Quadratics

You have determined inverses of linear functions by reflecting a function across the line $y = x$. Consider the basic quadratic function $f(x) = x^2$.

1. Use patty paper to reflect $f(x)$ across the line $y = x$ to graph its inverse.



2. Explain why the inverse is not a function based on its graph.

3. What is the domain and range of the function? What is the domain and range of the inverse of the function?

You can determine the equation of the inverse of the basic quadratic function $f(x) = x^2$ the same way you determined the equation of the inverse of a linear function.

Worked Example

$$f(x) = x^2$$

Step 1: Replace $f(x)$ with y. $\quad y = x^2$

Step 2: Switch the x and y variables. $\quad x = y^2$

Step 3: Solve for y. $\quad \pm\sqrt{x} = y$

So, the equation of the inverse is $y = \pm\sqrt{x}$.

Determining the equations of the inverses of exponential functions is a bit more complicated, and will be explored later in this course.

4. Explain why the inverse is not a function based on its equation.

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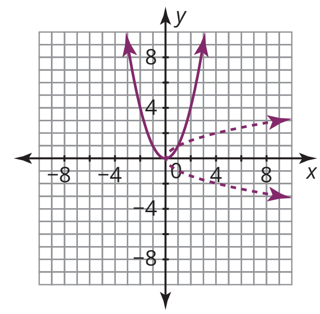
Answers

1.



2. The inverse is not a function because it does not pass the Vertical Line Test.
3. For the function:
   Domain: all real numbers
   Range: $y \geq 0$

   For the inverse of the function:
   Domain: $x \geq 0$
   Range: all real numbers
4. The inverse is not a function because the square root of a number can have two values, the positive square root and the negative square root.

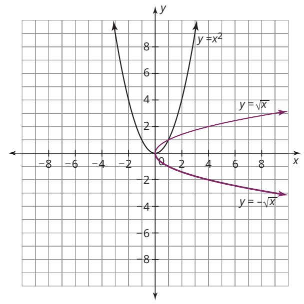
Answers

5a. Given $y = x^2$
Domain: $x \geq 0$
Range: $y \geq 0$
$y = \sqrt{x}$
Domain: $x \geq 0$
Range: $y \geq 0$
Given $y = x^2$
Domain: $x \leq 0$
Range: $y \geq 0$
$y = -\sqrt{x}$
Domain: $x \geq 0$
Range: $y \leq 0$

5b. The domain and range of the inverse are the reverse of the domain and the range of the original function.

5c. Yes. All the graphs represent functions.

You know that the inverse of $f(x) = x^2$ is not a function. However, you can *restrict the domain* of this function so that the inverse is also a function. To **restrict the domain** of a function means to define a new domain for the function that is a subset of the original domain.



5. Consider the graph of $f(x) = x^2$ and the graphs of the two equations that represent its inverse.

a. Identify the restrictions of $f(x) = x^2$ to produce the inverse equations $y = \sqrt{x}$ and $y = -\sqrt{x}$. Then state the domain and range of each inverse.

Restrictions for $y = x^2$ | **$y = \sqrt{x}$**

Domain: __________ **Domain:** __________

Range: __________ **Range:** __________

Restrictions for $y = x^2$ | **$y = -\sqrt{x}$**

Domain: __________ **Domain:** __________

Range: __________ **Range:** __________

b. How does the domain and range of the inverse relate to the restricted domain and range of the original function?

c. Do all the graphs represent functions? Explain your reasoning.

ACTIVITY **3.4**

More with Inverses of Quadratics

Marissa is competing in the Egg Drop Competition at her school's Science Fair. Competitors in the Egg Drop Competition are required to create a container in which they place a raw egg, and then drop the container from various heights to see if the egg breaks. The winner of the contest is the person whose container is dropped from the greatest height without breaking the egg.

Marissa is testing a container she built for the competition. She placed an egg in her container and dropped it from the roof of a building. The height of the egg can be modeled by the function $f(x) = -16x^2 + 64$, where x represents the time in seconds.

1. Define the independent and dependent quantities of $f(x)$.

2. What is the domain and range of $f(x)$ based on its equation?

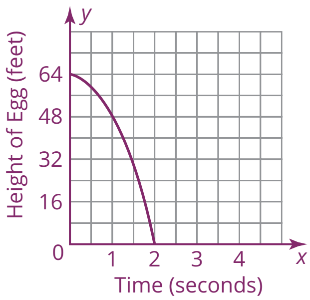
3. Determine any restrictions on the domain of $f(x)$ based on this problem situation. Explain your reasoning.

Answers

1. The independent quantity is the time in seconds, and the dependent quantity is the height of the egg in feet.
2. Domain: all real numbers
   Range: $y \leq 64$
3. Based on the problem situation, the domain must be $x \geq 0$, because only positive values for time make sense.

Answers

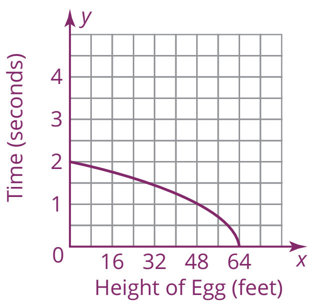
4.



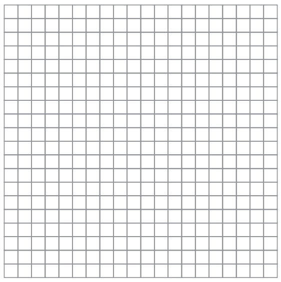
5. The independent quantity is the height of the egg in feet, and the dependent quantity is the time in seconds.

6. Domain: $x \leq 64$
Range: $y \geq 0$

7.



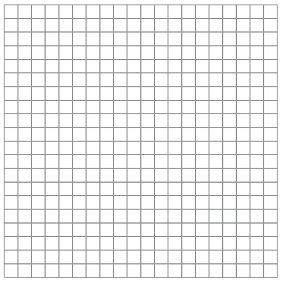
4. Graph $f(x) = -16x^2 + 64$ with the restricted domain based on this problem situation. Be sure to label your graph.



5. Define the independent and dependent quantities of the inverse of $f(x)$.

6. What is the domain and range of the inverse of $f(x)$ with the restricted domain?

7. Graph the inverse of $f(x)$ with the restricted domain. You may use different bounds than you used in Question 4.



8. Explain why the inverse of *f*(*x*) with the restricted domain is a function. Then, write an equation for the inverse.

9. Explain what the inverse models in terms of this problem situation.

10. After 1.5 seconds, what is the egg's height? Explain which function you used and how you determined your answer.

11. After how many seconds is the egg at a height of 55 feet? Explain which function you used and how you determined your answer.

Answers

8. The inverse of $f(x)$ with the restricted domain is a function because it passes the Vertical Line Test.

   $y = \pm\sqrt{\frac{64 - x}{16}}$ or

   $y = \pm\sqrt{\frac{x - 64}{-16}}$

   The original function has a restricted domain, so the inverse includes only the positive square root:

   Inverse of $f(x) = \sqrt{\frac{64 - x}{16}}$

   or

   Inverse of $f(x) = \sqrt{\frac{x - 64}{-16}}$.

9. The inverse of the function models the time that the egg is in the air in seconds based on its height.

10. After 1.5 seconds, the egg's height is 28 feet. I substituted the value 1.5 into the function $f(x)$.

    $f(x) = -16x^2 + 64$

    $f(1.5) = -16(1.5)^2 + 64$

    $f(1.5) = 28$

11. The egg's height is 55 feet at 0.75 seconds. I substituted the value 55 into the function $f^{-1}(x)$.

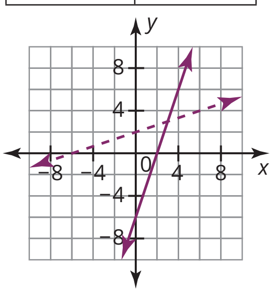
    $f^{-1}(55) = \frac{3}{4} = 0.75$

Answer

1a.

x	$f(x)$
−2	−12
−1	−9
0	−6
1	−3
2	0

Inverse of $f(x)$	
x	y
−12	−2
−9	−1
−6	0
−3	1
0	2



$y = 3x - 6$

$x = 3y - 6$

$x + 6 = 3y$

$\frac{1}{3}x + 2 = y$

Yes. The function $f(x)$ is a one-to-one function because both $f(x)$ and the inverse of $f(x)$ are functions.

ACTIVITY 3.5

One-to-One Functions

In this activity, you will determine the inverse of a function using multiple representations.

1. For each given function, determine the inverse using each representation.

- **Complete a table of values for the function and its inverse.**
- **Sketch the graph of the function using a solid line. Then sketch the inverse of the function on the same coordinate plane using a dashed line.**
- **Write an equation for the inverse.**
- **Determine whether the function is a one-to-one function. Explain your reasoning.**

A function is a **one-to-one function** if both the function and its inverse are functions.

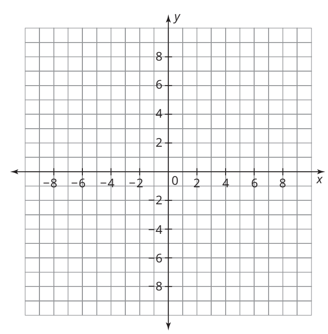
Remember:

Use a straightedge to draw your lines.

a. $f(x) = 3x - 6$

x	$f(x)$
−2	
−1	
0	
1	
2	

Inverse of $f(x)$	
x	y
	−2
	−1
	0
	1
	2



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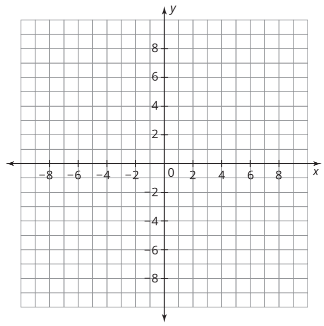
b. $g(x) = -x + 4$

x	$g(x)$
−2	
−1	
0	
1	
2	

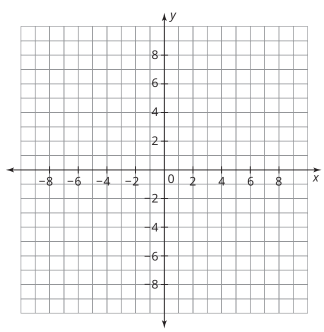
Inverse of $g(x)$	
x	y
	−2
	−1
	0
	1
	2



c. $h(x) = 2$

x	$h(x)$
−2	
−1	
0	
1	
2	

Inverse of $h(x)$	
x	y
	−2
	−1
	0
	1
	2

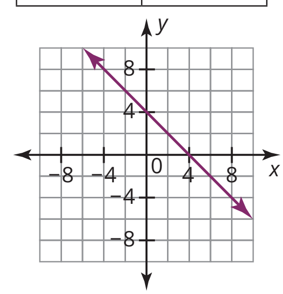


Answers

1b.

x	$g(x)$
−2	6
−1	5
0	4
1	3
2	2

Inverse of $g(x)$	
x	y
6	−2
5	−1
4	0
3	1
2	2



$y = -x + 4$

$x = -y + 4$

$x - 4 = -y$

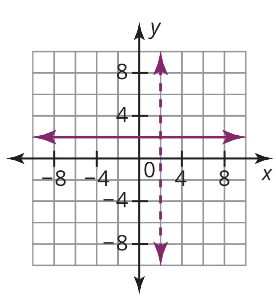
$-x + 4 = y$

Yes. The function $g(x)$ is a one-to-one function because both $g(x)$ and the inverse of $g(x)$ are functions.

1c.

x	$h(x)$
−2	2
−1	2
0	2
1	2
2	2

Inverse of $h(x)$	
x	y
2	−2
2	−1
2	0
2	1
2	2



$y = 2$

$x = 2$

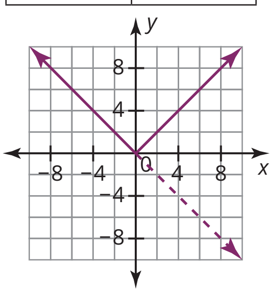
No. The function $h(x)$ is not a one-to-one function because the inverse of $h(x)$ is not a function.

Answers

1d.

x	$r(x)$
−2	2
−1	1
0	0
1	1
2	2

Inverse of $r(x)$	
x	y
2	−2
1	−1
0	0
1	1
2	2



$y = |x|$

$x = |y|$

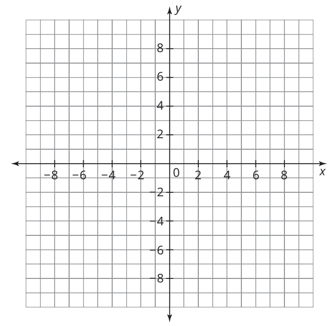
No. The function $r(x)$ is not a one-to-one function because the inverse of $r(x)$ is not a function.

1e.

x	y
−2	8
−1	5
0	4
1	5
2	8

d. $r(x) = |x|$

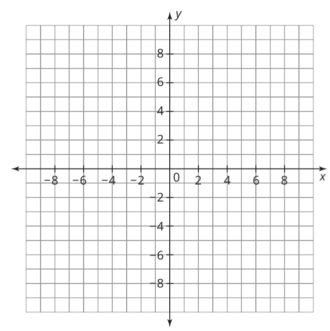
x	$r(x)$
−2	
−1	
0	
1	
2	



Inverse of $r(x)$	
x	y
	−2
	−1
	0
	1
	2

e. $s(x) = x^2 + 4$

Inverse of $r(x)$	
x	y
−2	
−1	
0	
1	
2	



Inverse of $s(x)$	
x	y
	−2
	−1
	0
	1
	2

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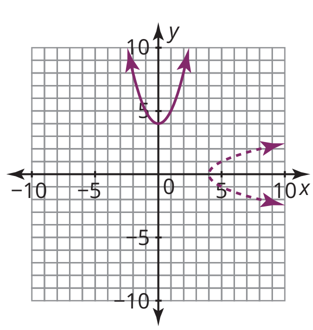
Inverse of $s(x)$	
x	y
8	−2
5	−1
4	0
5	1
8	2



$y = x^2 + 4$

$x = y^2 + 4$

$x - 4 = y^2$

$\pm\sqrt{x - 4} = y$

No. The function $s(x)$ is not a one-to-one function because the inverse of $s(x)$ is not a function.

2. **How can you determine whether an inverse exists given a linear function?**

3. **Can a linear function and its inverse be the same function? If so, provide an example. If not, explain why not.**

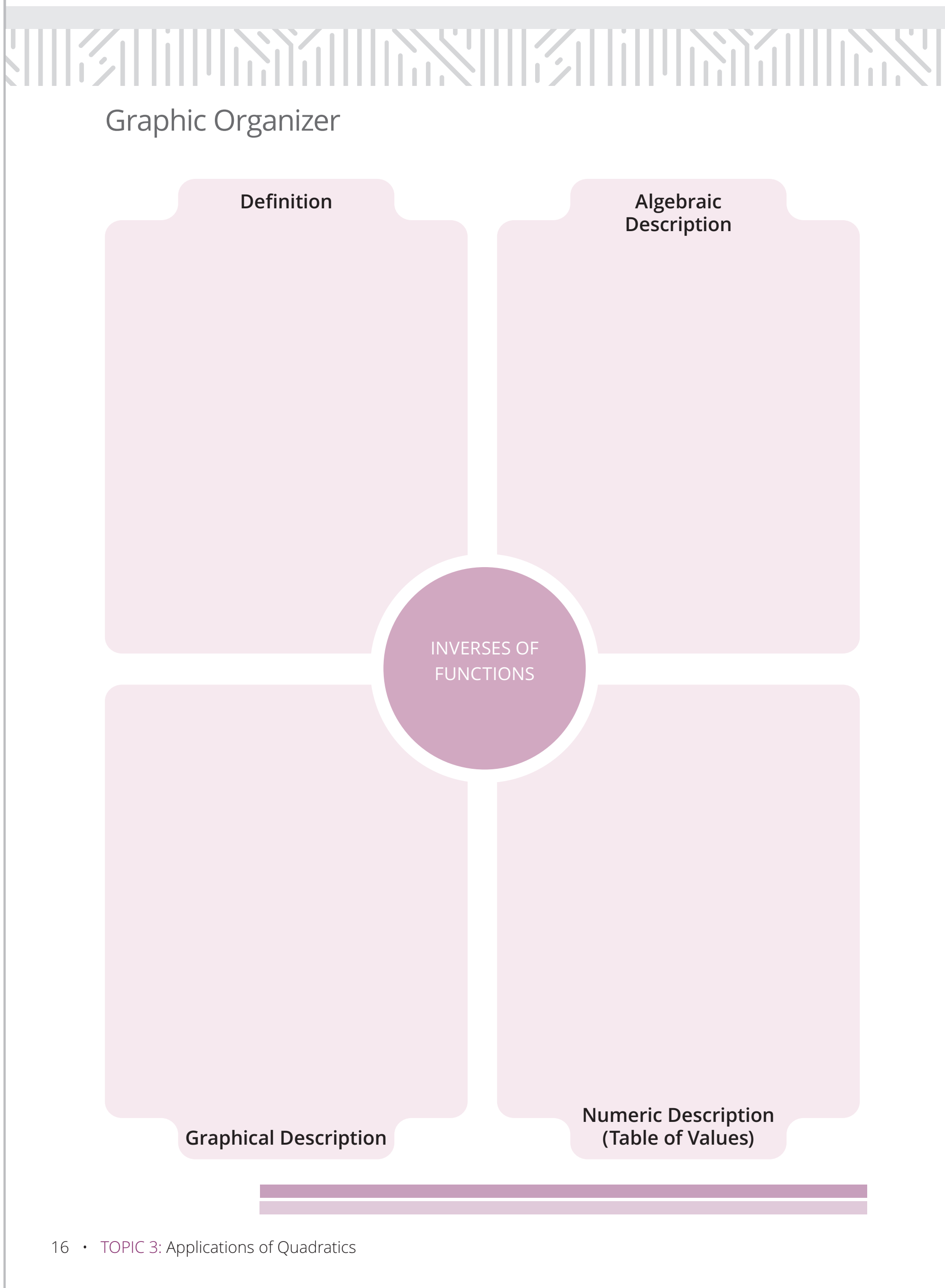
4. **Complete the graphic organizer on the next page. Write the definition for the inverse of a function. Then describe how to determine the inverse of a function algebraically, graphically, and numerically.**

NOTES

For a one-to-one function $f(x)$, the notation for its inverse is $f^{-1}(x)$. The notation for inverse, $f^{-1}(x)$, does not mean the same thing as x^{-1}. The expression x^{-1} can be rewritten as $\frac{1}{x}$; however, $f^{-1}(x)$ cannot be rewritten, because it is only used as notation. In other words, $f^{-1}(x) \neq \frac{1}{f(x)}$.

Answers

2. Sample answer.
   The only linear functions for which an inverse does not exist are constant functions. For all other linear functions, an inverse exists.
3. Yes. The inverse of any function that is symmetric about the line $y = x$ will be equal to the function. For example, for the function $f(x) = x$, the inverse is $f^{-1}(x) = x$. Also, for the function $f(x) = -x + 1$, the inverse is $f^{1}(x) = -x + 1$.
4. Definition: An inverse of a function takes the output value, performs operations on this value, and arrives back at the original function's input value.
   Algebraic: To determine the inverse of a function using algebra, switch the x and y variables in the equation, and solve for y.
   Graphic: To determine the inverse of a graph of a function, reflect the original graph across the line $y = x$.
   Numeric: To determine the inverse of a table of values, switch the independent and dependent values.


Graphic Organizer

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TALK the TALK

1-2-1

In this lesson, you determined the inverses of linear and quadratic functions. You also determined whether the inverses were also functions.

Recall that a function is a one-to-one function if both the function and its inverse are functions.

1. **Adam and Stacey are working on a homework assignment for which they must identify all functions that are one-to-one functions. Adam says that all linear functions are one-to-one functions, so they don't even need to look at the linear functions. Stacey disagrees, and says that not all linear functions are one-to-one functions. Who is correct? Explain how you determined which student is correct.**

2. **Complete each sentence with *always*, *sometimes*, or *never*.**

   a. **A linear function is ____________ a one-to-one function.**

   b. **An exponential function is ____________ a one-to-one function.**

   c. **A quadratic function is ____________ a one-to-one function.**

   d. **A linear absolute value function is ____________ a one-to-one function.**

NOTES

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Answers

1. Stacey is correct. Not all linear functions are one-to-one functions. The inverse of any horizontal line is a vertical line, and vertical lines are not functions.

2a. sometimes

2b. always

2c. never

2d. never