4

Modeling Behavior

Using Quadratic Functions to Model Data

MATERIALS

Graphing technology
Patty paper

Lesson Overview

Students begin the lesson by determining a quadratic regression equation to model a set of data and use the regression equation to make predictions. Next, they are given a quadratic equation that models a context, but this time students see the need for an inverse equation because they must solve for the independent variable when the dependent variable is provided. Throughout the lesson, students identify the independent and dependent quantities and domain and range of functions in order to make sense of an inverse of function.

Algebra 2

Attributes of Functions and Their Inverses

(2) The student applies mathematical processes to understand that functions have distinct key attributes and understand the relationship between a function and its inverse. The student is expected to:

(C) describe and analyze the relationship between a function and its inverse (quadratic and square root, logarithmic and exponential), including the restriction(s) on domain, which will restrict its range.

Quadratic and Square Root Functions, Equations, and Inequalities

(4) The student applies mathematical processes to understand that quadratic and square root functions, equations, and quadratic inequalities can be used to model situations, solve problems, and make predictions. The student is expected to:

(E) formulate quadratic and square root equations using technology given a table of data.

Number and Algebraic Methods

(7) The student applies mathematical processes to simplify and perform operations on expressions and to solve equations. The student is expected to:

(I) write the domain and range of a function in interval notation, inequalities, and set notation.

Data

(8) The student applies mathematical processes to analyze data, select appropriate models, write corresponding functions, and make predictions. The student is expected to:

(A) analyze data to select the appropriate model from among linear, quadratic, and exponential models.

(B) use regression methods available through technology to write a linear function, a quadratic function, and an exponential function from a given set of data.

(C) predict and make decisions and critical judgments from a given set of data using linear, quadratic, and exponential models.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 5.B, 5.F, 5.G

Essential Ideas

- Some data in context can be modeled by a quadratic regression equation. The regression equation can be used to make predictions; however there may be limitations on the domain depending on the context.
- When a problem requires using a given function to determine the independent quantity when a dependent quantity is given, determining the inverse of the original function may be a more efficient way to handle the situation.

Lesson Structure and Pacing: 1 Day

Engage

Getting Started: That Might Be a Bad Idea . . .

Students analyze a table of values related to a context and create a scatter plot to represent the data.

Develop

Activity 4.1: Using Quadratic Functions to Model Data

Students use the data from the Getting Started to determine the regression equation. The regression equation is used to identify key characteristics of the function, interpret them with respect to the problem situation, and make predictions.

Activity 4.2: Analyzing a Quadratic Model and Its Inverse

This activity uses a context to develop the need for an inverse of a function. Students complete a table for a given quadratic function. They answer questions focused on the independent and dependent quantities and the domain and range, and sketch a graph representing the function. Students are then given a value for the dependent quantity, and asked to estimate and calculate exactly its corresponding independent quantity. They describe how a function that reverses the independent and dependent quantities would answer this question more efficiently.
Students sketch a graph of the new function by reversing coordinate pairs, and use their graph to answer additional questions.

Demonstrate

Talk the Talk: Living to 100

Students analyze a table of values related to a context and create a scatter plot. Next students use technology to calculate the linear, exponential, and quadratic regression equations and determine which best represents the problem situation. Students use the regression question that best models the data to make a prediction.

Getting Started: That Might Be a Bad Idea . . .

Facilitation Notes

In this activity, students analyze a table of values related to a context and create a scatter plot to represent the data.

Ask a student to read the introduction aloud. Discuss as a class.

Questions to ask

- Did you ever place a can of soda in the freezer for too long? What happened to it?
- Why does a soda can burst if left in the freezer for too long?
- Why does the soda take up more space when it is frozen?
- What is volume?

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Questions to ask

- As the temperature decreases, why does the volume decrease then increase?
- At what temperature in the table does the soda volume begin to increase as the temperature decreases? Is there any special meaning to this temperature?
- Why does the volume increase at high temperatures? At low temperatures?
- Does the data appear to be linear? Why not?
- Does the data appear to have an absolute minimum or maximum? If so, which one?
- What types of functions have an absolute minimum or maximum?
- Why can you rule out the possibility that this data can be modeled by an absolute value function?
- What type of function is suggested by the scatter plot?
- Was it easier to notice trends from the table or the graph? Explain why.
- What are the components of the modeling process?
- Which components of the modeling process are highlighted in this activity?
- What actions did you complete that applied to those steps in the modeling process?

Differentiation strategies

To extend the lesson, have students investigate the science behind this data trend. Pose additional questions, such as:

- Does the volume ever reach a maximum?
- Does the volume reach a minimum at a specific temperature?
- Does the data change for other liquids? If so, how?

Summary

A scatter plot can be created from a table of data to help recognize the function that best models the data.

Activity 4.1

Using Quadratic Functions to Model Data

DEVELOP

Facilitation Notes

In this activity, students use the data from the Getting Started to determine the regression equation. The regression equation is used to identify key characteristics of the function, interpret them with respect to the problem situation, and make predictions.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

As students work, look for

- Errors in the a-value of the quadratic regression equation, writing it as 8.8 rather than 0.00088. Depending on the technology used, the a-value of the quadratic equation may be expressed as $a = 8.8263492\text{E-}4$. Students may need to be reminded how to interpret this value, first as $a = 8.8263492 \times 10^{-4}$, then as 0.00088.
- Inequality or interval notation.

Differentiation strategies

To assist all students,

- Discuss what is a reasonable number of decimal places when writing the regression equation. For example, a steadfast rule of two decimal places may make sense in most cases, but not when $a = 0.00088$.
- Clarify the difference between how exact the equation should be when *writing* the regression equation versus *using* the regression equation for calculations. When making calculations, it makes sense to use all the decimal places provided by the technology so that answers will be as exact as possible, then have students round the answers as desired. Demonstrate how to transfer the a-, b-, and c-values from the regression process to the $y =$ component using technology. This will also make it easier for students to compare answers.

Questions to ask

- What function did you choose to model this data?
- What is the r^2-value for your regression equation? What does this imply about the fit of your data?

- How do the values from the table generated by the regression equation compare to those in the table of original data?
- Why did you need to know the minimum to calculate the range?
- What is the minimum? What does it mean with respect to the problem situation?
- How did you determine the *y*-intercept?
- If $x = 0$, what does this mean with respect to the problem situation?
- When the temperature is 0°F, what is the volume of the soda?
- If $y = 0$, what does this mean with respect to the problem situation?
- Could the volume ever be 0 cm^3?
- Which components of the modeling process are highlighted in these questions?
- What actions did you complete that applied to those steps in the modeling process?

Have students work with a partner or in a group to complete Question 4. Share responses as a class.

Questions to ask

- How did you make your predictions?
- Did you use the graph, the regression equation, or the table to determine the temperatures?
- Do you think your predictions are relatively accurate? Why or why not?

Have students work with a partner or in a group to complete Question 5. Share responses as a class.

Questions to ask

- How did you describe the table of data? The scatter plot?
- What key characteristics of the equation and graph did you notice?
- How did you explain why you selected a quadratic equation to model the data?
- In your model, is the input the volume or the temperature? What is the output?
- Does your model make unrealistic predictions for extremely high or low temperatures?
- What domain do you think makes sense for your regression equation?

Summary

Some data in context can be modeled by a quadratic regression equation. The regression equation can be used to make predictions; however there may be limitations on the domain depending on the context.

Activity 4.2
Analyzing a Quadratic Model and Its Inverse

Facilitation Notes

This activity uses a context to develop the need for an inverse of a function. Students complete a table for a given quadratic function. They answer questions focused on the independent and dependent quantities and the domain and range and sketch a graph representing the function. Students are then given a value for the dependent quantity and asked to estimate and calculate exactly its corresponding independent quantity. They describe how a function that reverses the independent and dependent quantities would answer this question more efficiently. Students sketch a graph of the new function by reversing coordinate pairs and use their graph to answer additional questions.

Ask a student to read the introduction aloud. Discuss as a class.

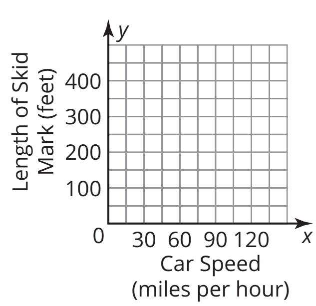
Questions to ask

- What causes vehicles to leave skid marks?
- Do you think faster cars leave longer or shorter skid marks? Why?
- How are stopping distance and length of skid marks related?
- Give an example of how weather would affect stopping distances and the length of skid marks left by a vehicle.
- How would the road surface, the grade of the road, or the vehicle type affect stopping distances and length of skid marks left by a vehicle?
- Do you think if a car increased its speed by 20 mph that the length of its skid marks would also increase by 20 feet? Why or why not?

Have students work with a partner or in a group to complete Questions 1 through 4. Share responses as a class.

Differentiation strategy

To scaffold support, provide a scale for the graph in Question 3.



Questions to ask

- Does the length of the skid mark depend on the speed of the car, or does the speed of the car depend on the length of the skid mark?
- Why is the graph only increasing?

- How does the domain and range of the function compare to the domain and range of the problem situation?
- Did you use the graph or table to estimate the speed of the car?
- Why can't the graph be used to determine the exact speed of the car?
- How did you determine the exact speed of the car?
- Is the estimate close to the exact speed of the car?

Have students work with a partner or in a group to complete Questions 5 through 8. Share responses as a class.

Differentiation strategy

To scaffold support, suggest that students make a table of values first, then refer to the table of values to sketch the graph of the inverse of a function.

Questions to ask

- Why would it be helpful to have a new function?
- When the length of a skid mark becomes the independent variable and the speed of the vehicle becomes the dependent variable, what changes on the graph?
- What is the relationship between the domain and range of the original function and the domain and range of the inverse of the function?
- How did you label the axes on the graph of the new function?
- How did you locate the points on the graph of the new function?
- Is this new graph increasing or decreasing?
- How does the shape of this graph compare to the shape of the original graph?

Summary

When the context requires using a given function to determine the independent quantity when a dependent quantity is given, determining the inverse of the original function may be a more efficient way to handle the situation.

DEMONSTRATE

Talk the Talk: Living to 100

Facilitation Notes

In this activity, students analyze a table of values related to a context and create scatter plot. Next, students use technology to calculate the linear, exponential, and quadratic regression equations and determine which best represents the problem situation. Students use the regression question that best models the data to make a prediction.

Have students work with a partner or in a group to answer Questions 1 through 4. Share responses as a class.

As students work, look for

- Students using the given values for *x* instead of letting *x* be the number of years since 1994.

Questions to ask

- What function did you choose to model this data?
- What is the r^2-value for your regression equation? What does this imply about the fit of your data?
- Why does it make sense to use 0 instead of 1994 when determining the regression equation?
- If $x = 0$, what does this mean in terms of the problem situation?

Summary

A quadratic regression equation can be used to model data and make predictions.

NOTES

Warm Up Answer

$y = 1.2x + 31.333$

4

Modeling Behavior

Using Quadratic Functions to Model Data

Warm Up

Determine a linear regression equation that best models the data.

x	y
1	32
2	35
3	34
4	35
5	39
6	38
7	40
8	42
9	41

Learning Goals

- Use a quadratic function to model data.
- Interpret characteristics of a quadratic function in terms of a problem situation.
- Use graphs of quadratic functions to make predictions.
- Interpret the inverse of a function in terms of a problem situation.

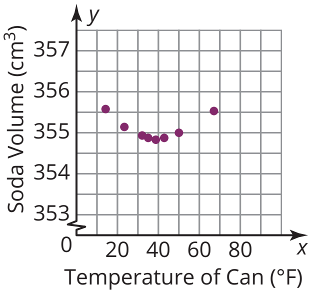
You know how to model data with regression equations and how to write inverses of linear and quadratic functions. How can you determine whether a quadratic regression equation may best model the data?

LESSON 4: Modeling Behavior • 1

Answers

1. As the temperature decreases, the volume decreases and then increases.
2.

Temperature and Volume of a Soda Can in a Freezer



GETTING STARTED

That Might Be a Bad Idea. . .

A 12-ounce can of soda was put into a freezer. The table shows the volume of the soda in the can, measured at different temperatures.

The first step of the modeling process is to notice and wonder. What do you notice about the data? Is there a question it brings to mind that you wonder about?

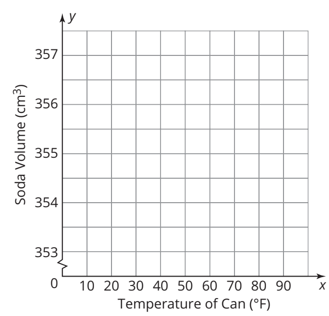
Temperature of Can (°F)	Soda Volume cm³
68.0	355.51
50.0	354.98
42.8	354.89
39.2	354.88
35.6	354.89
32.0	354.93
23.0	355.13
14.0	355.54

1. Describe the data distribution.

2. Create a scatter plot of the data. Sketch the plot of points on the coordinate plane shown.

The second step of the modeling process is to organize and mathematize. The scatter plot is a way to organize the data.

Temperature and Volume of a Soda Can in a Freezer



ACTIVITY 4.1

Using Quadratic Functions to Model Data

Let's continue to analyze the data and make some predictions about the volume of soda at different temperatures.

1. Use technology to calculate the regression equation that best models the data in the previous activity. Sketch the graph of the regression equation on the coordinate plane on which you created your scatter plot. Explain why the regression equation best models the data.

You can mathematize the data by modeling it with an appropriate regression equation.

2. State the domain and range of your function. How do they compare to the domain and range of this problem situation?

3. Use the regression equation to answer each question.

a. Determine the *y*-intercept and interpret its meaning in terms of this problem situation.

The third step of the modeling process is to predict and analyze, and the fourth step is to test and interpret. These questions focus on these two steps of the process.

b. Determine the *x*-intercepts, and interpret the meaning of each in terms of this problem situation.

LESSON 4: Modeling Behavior • 3

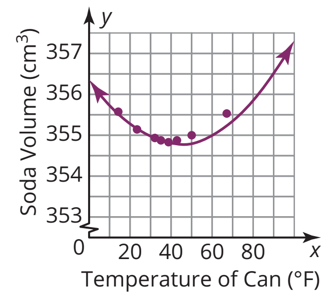
3a. The y-intercept is (0, 356.35). This is the volume of the soda when the temperature is 0°F.

3b. There are no x-intercepts. The volume of the soda will never be 0 cm^3.

Answers

1. Let x represent the temperature of the can in degrees Fahrenheit. Let y represent the volume of the soda in cm^3.
$y = 0.00088x^2 - 0.07223x + 356.35$

Temperature and Volume of a Soda Can in a Freezer



The quadratic regression equation is the best fit for the data because the parabola passes through almost all the points. The r^2-value for the quadratic regression fit is 0.994, which is very close to 1.

2. For the function:

Domain: $-\infty < x < \infty$

Range: $y \geq 354.88$

For the problem situation:

Domain: $14 \leq x \leq 68$

Range: $354.88 \leq y \leq 355.54$

The domain of the function is all possible values, and the range is all values greater than or equal to 354.88. The domain and range of the problem situation is restricted to the spread of the values in the table.

Answers

4a. The volume of the soda is 355.26 cm^3 at 20°F.

4b. The volume of the soda is 355.2 cm^3 at 60°F.

5. Sample answer. Data was provided for the different temperatures of a soda can and the corresponding volume of the soda can. Looking at the table, I could see that as the temperature decreased, the volume decreased until the temperature reached 39.2°. When the temperature reached 39.2° or below, the volume began to increase. Because there is a minimum volume, I thought the function modeling the data would be quadratic, and after completing a regression of the data, I was correct. The regression equation is $y = 0.00088x^2 - 0.07223x + 356.35$ with an r^2-value of 0.994. I know the can will never have a volume of 0 cm^3, and I know it will not increase infinitely in volume either. Without more data, I am only confident that my equation works for the specified values in the table, with a domain (temperature in degrees F) of $14 \le x \le 68$ and a range (volume in cm^3) of $354.88 \le y \le 355.54$.

4. Predict the volume of the soda can when the temperature is:

a. 20°F.

b. 60°F.

5. Write a summary of the problem situation, your model as the solution, and any limitations of your model.

4 • TOPIC 3: Applications of Quadratics

ELL Tip

Discuss the root word of *limitations*, which is *limit*. As a noun, a *limit* is *a restriction on the amount of something*. Connect the meaning with the phrase *speed limit*. In the context of the Question 5, the question is asking whether students would like to make restrictions on their model; perhaps their model makes accurate predictions for the context only in some cases or for limited temperatures. Help students rephrase the directions to Question 5 in their own words.

ACTIVITY
4.2

Analyzing a Quadratic Model and Its Inverse

Arlen City Police Department is offering special classes for interested high school students this summer. Elsa decides to enroll in an introductory forensic science class. On the first day, Dr. Suarez tells Elsa's class that crime scenes often involve speeding vehicles which leave skid marks on the road as evidence. Taking into account the road surface, weather conditions, the percent grade of the road, and vehicle type, they use this function:

$$f(s) = 0.034s^2 + 0.96s - 26.6$$

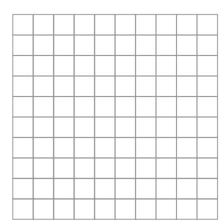
to determine the length in feet of skid marks left by a vehicle based on its speed, s, in miles per hour.

1. Complete the table based on $f(s)$. Label the column titles with the independent and dependent quantities and their units.

25	
30	
45	
55	
60	
75	
90	
100	
110	

2. According to the table, what are the domain and range for the problem situation?

3. Graph the table values and sketch the graph of $f(s)$ on the grid shown. Label the axes.



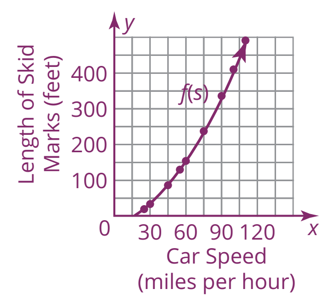
Answers

1.

Speed of the Vehicle (miles per hour)	Length of Skid Marks (feet)
25	18.65
30	32.8
45	85.45
55	129.05
60	153.4
75	236.65
90	335.2
100	409.4
110	490.4

2. Domain: $25 \leq x \leq 110$
Range: $18.65 \leq y \leq 490.4$

3.



ELL Tip

Three non-mathematical terms that appear in this activity are *forensic science, skid marks*, and *grade of a road*. Define *forensic science* as *the application of scientific principles to matters of criminal justice, specifically relating to physical evidence*. Define *skid marks* as *long black marks left on a road surface by the tires of a skidding vehicle* and provide an illustration. Define the *grade of a road* as *the steepness, or slope, of a road*. Explain why a forensic scientist may be interested in analyzing skid marks.

Answers

4a. Estimating from the data table, the car was traveling at about 80 miles per hour.

4b. Using the Quadratic Formula or the table generated by $f(s)$, the car was traveling at 84.9 miles per hour.

5. The new function would be the inverse of the original function. The length of the skid mark would be the independent quantity and the speed of the vehicle would be the dependent quantity. The domain would be $18.65 \leq x \leq 490.4$, and the range would be $25 \leq y \leq 110$.

During another class period, Dr. Suarez takes Elsa's class to a mock crime scene to collect evidence.

4. One piece of evidence is a skid mark that is 300 feet long.

a. Use the graph to estimate the speed of the vehicle that created this skid mark. Explain your process.

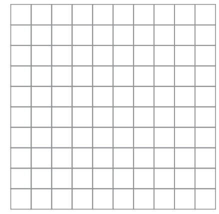
Ask yourself:

How do these data differ from the data in the table?

b. Determine the exact speed of the vehicle that created this skid mark. Show your work.

5. Describe a new function that Elsa can use to determine the speed of a vehicle given the length of a skid mark it created. In your description, include information about the independent and dependent variables, and the domain and range of this problem situation.

6. Predict what you think the graph of the new function will look like and sketch the graph on the grid shown.



7. Use your graph to estimate the car's speed before stopping for each given skid mark length.

a. 50 feet

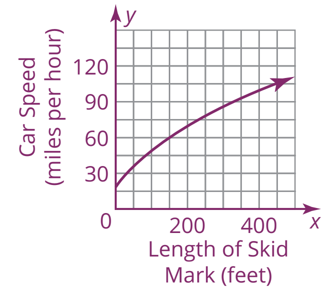
b. 175 feet

c. 350 feet

8. Write a report about the length of skid marks left by vehicles and vehicle speeds. Discuss possible factors that would affect the length of the skid marks left by a vehicle, and what effect these factors would have on the graph of *f*(*s*) and the graph of its inverse.

Answers

6.



7a. For 50 feet, the estimated speed is 35 miles per hour.

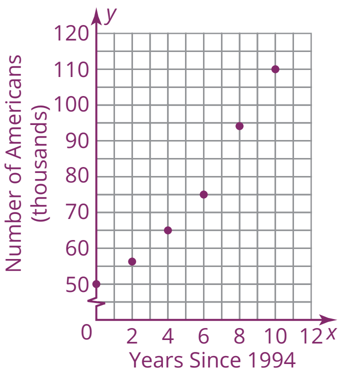
7b. For 175 feet, the estimated speed is 64 miles per hour.

7c. For 350 feet, the estimated speed is 92 miles per hour.

8. Answers will vary. There is a quadratic relationship between the speed of a car and the length of the skid marks it may leave on a road surface. The quadratic function may vary depending upon factors such as road surface, weather, percent grade of the road, vehicle type, etc. For example, if the road is slick or the weather is icy, the values in the function may change to recognize the fact that for a given temperature, the length of the skid marks would be greater than the results from the original quadratic function. The results would be the opposite in an inverse of the function for slick roads or icy weather; for a given length of skid mark, the speed would be lower than the results from the original quadratic function.

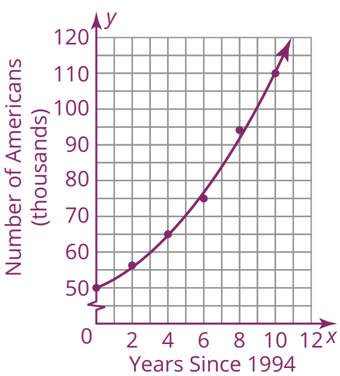
Answers

1.



2. Let x represent the number of years since 1994. Let y represent the number of Americans, in thousands, who lived to be over 100 years old.
Linear Regression: $y = 6.06x + 44.7$, $r^2 = 0.9612$
Exponential Regression: $y = 48.23 \cdot 1.08^x$, $r^2 = 0.9899$
Quadratic Regression: $y = 0.40x^2 + 2.04x + 50.0$, $r^2 = 0.9973$

The quadratic regression equation is the best fit for the data because the parabola passes through almost all the points. The r^2-value or the quadratic regression fit is 0.997, which is very close to 1.



NOTES

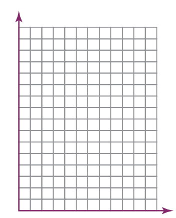
TALK the TALK

Living to 100

Using data from the US Census Bureau, the table lists the number of Americans, in thousands, who lived to be over 100 years old for the specified years.

Year	Number of Americans (thousands)
1994	50
1996	56
1998	65
2000	75
2002	94
2004	110

1. **Graph the table values on the grid shown. Let *x* be the number of years since 1994. Label the axes.**



2. **Use technology to calculate the linear, exponential and quadratic regression equations. Which model best fits the data? Explain your reasoning. Sketch the graph of the best fit regression equation on the coordinate plane on which you created your scatter plot.**

3. **State the domain and range of your function. How do they compare to the domain and range of this problem situation?**

4. **Use your regression equation to predict the number of Americans that will live to be over 100 years old in the year 2025.**

8 • TOPIC 3: Applications of Quadratics

3. For the function:
Domain: $-\infty < x < \infty$
Range: $y \geq 50$
For the problem situation:
Domain: $0 \leq x \leq 10$
Range: $50 \leq y \leq 110$

4. There will be approximately 499.405 thousand Americans that will live to be over 100 years old in the year 2025.