5

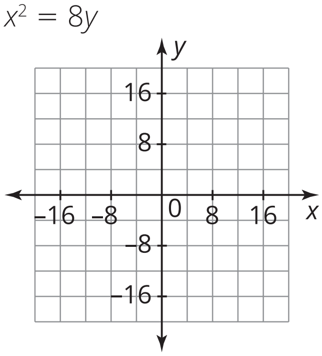
Going the Equidistance

Equation of a Parabola

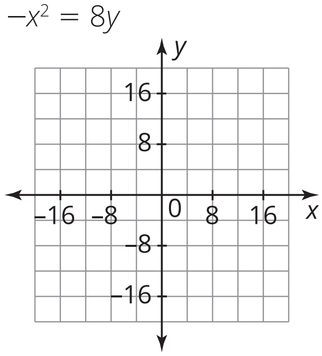
Warm Up

Use graphing technology to graph each equation and its inverse. Determine the vertex and axis of symmetry of each graph.

1. $x^2 = 8y$



2. $-x^2 = 8y$



Learning Goals

- Derive the equation of a parabola given the focus and the directrix.
- Solve problems using characteristics of parabolas.

Key Terms

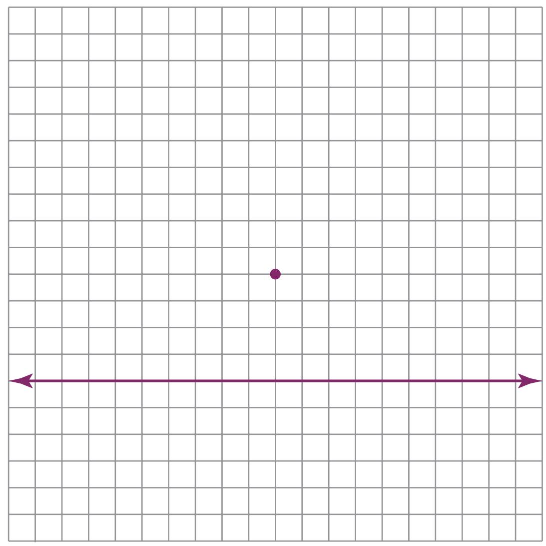
- locus of points
- focus
- directrix
- vertex of a parabola
- concavity
- general form of a parabola
- standard form of a parabola

You have studied parabolas as quadratic functions. What are the characteristics of a parabola determined by the set of points equidistant from both a given point and a given line?

GETTING STARTED

Hocus Pocus . . . Make the Locus!

Consider the grid with the plotted point and graphed line. Can you create a set of points that are the same distance from both the point and the line?



1. Construct a circle with a radius of 1 unit using the point as the circle's center.

a. What is the relationship between the points on the circle and the plotted point?

b. Are any of the points on the circle the same distance from the plotted point and the graphed line? Explain your reasoning.

2. Construct a circle with a radius of 2 units using the point as the circle's center.

a. **Are any of the points on the circle the same distance from the plotted point and the graphed line? Explain your reasoning.**

b. **Plot a point on the circle that is the same distance from the circle's center and the line.**

3. **Continue to construct a total of eight concentric circles using the point as the center with the radius of each successive circle increasing by 1 unit.**

   a. **How can you determine which points on each new circle are the same distance from the circle's center and the line?**

   b. **Plot the points on each circle that are the same distance from the circle's center and the line.**

Think about:

Will any of the points be below the graphed line?

4. **Connect the points you plotted with a smooth curve.**

   a. **What shape did you draw?**

   b. **What do all the points on the curve have in common?**

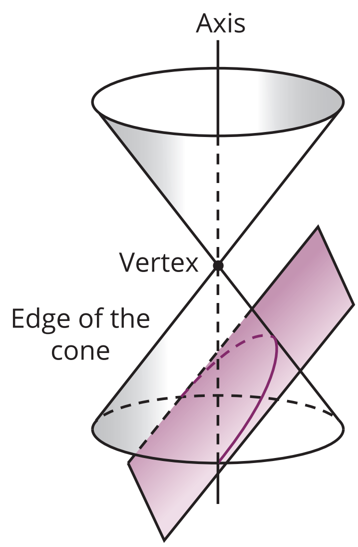
ACTIVITY

5.1

A Parabola as a Locus of Points

You previously studied parabolas as quadratic functions and recognized the equation of a parabola as $y = x^2$. You analyzed equations and graphed parabolas based on the position of the vertex and additional points determined by using x-values on either side of the axis of symmetry.

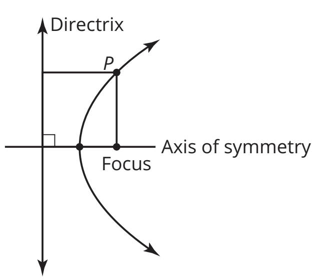
Recall that when a plane intersects one nappe of the double-napped cone parallel to the edge of the cone, the curve that results is a parabola.



Remember:

Everything you already know about parabolas remains true. The axis of symmetry of a parabola is a line that passes through the parabola and divides it into two symmetrical parts that are mirror images of each other.

In this lesson, you will explore a parabola as a *locus of points* to determine more information about the parabola. A **locus of points** is a set of points that share a property. A parabola is the set of all points in a plane that are equidistant from both a *focus* and a *directrix*. The **focus** is a point which lies inside the parabola on the axis of symmetry. The **directrix** is a line that is perpendicular to the axis of symmetry and lies outside the parabola and does not intersect the parabola.



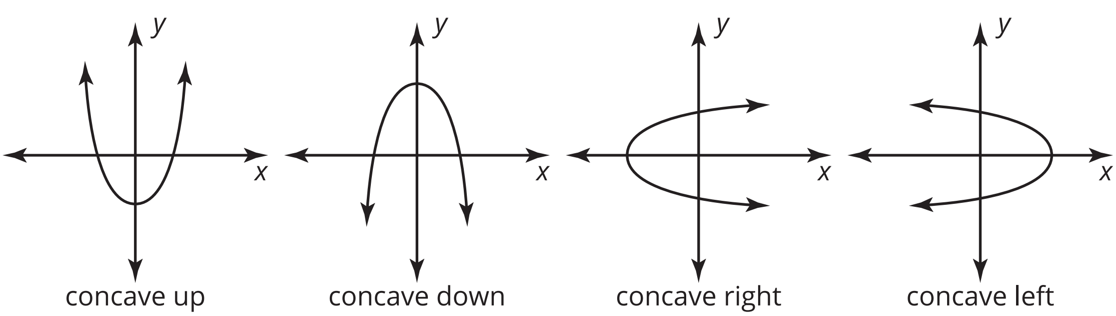
1. **Create a parabola by folding patty paper.**

   a. **Take a piece of patty paper. Near the bottom of the paper, draw a line (the directrix). Draw a point (the focus) above the line and label it point *F*.**

   b. **Fold the patty paper so that point *F* and the line meet and make a crease. Next, slide point *F* along the line continuing to crease the patty paper. Repeat this process for at least 30 points along the line.**

   c. **What conic section is formed by the folds? Outline the shape.**

The **concavity** of a parabola describes the orientation of the curvature of the parabola. A parabola can be concave up, concave down, concave right, or concave left, as shown. The **vertex of a parabola** is the point on the axis of symmetry which is exactly midway between the focus and the directrix. It is also the point where the parabola changes direction.



2. **Label the vertex of the parabola you constructed.**

3. Investigate the parabola you constructed using patty paper.

a. Draw a point on the directrix and label it point *D*. Draw a line from point *F*, the focus, to point *D*. Fold the paper so points *F* and *D* meet. What is the relationship between the line formed by the crease and $\overline{FD}$?

b. Draw a line perpendicular to the directrix through point *D*. Label the point where the perpendicular line intersects the crease as point *P*. Where does point *P* lie?

c. What is true about the distances from point *P* to point *F* and to point *D*? How do you know this is true?

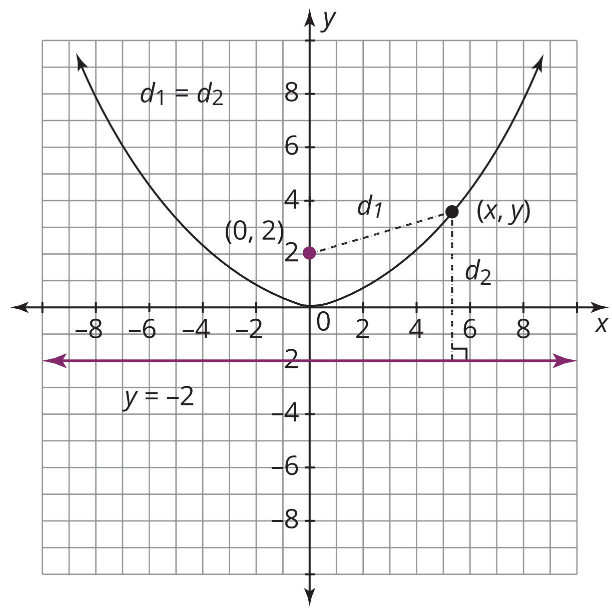
d. Draw two additional points on the directrix and repeat parts (a) through (c). What do you notice?

4. Summarize what you discovered about the distance from any point on a parabola to the focus and directrix.

ACTIVITY 5.2

Equations of a Parabola

The parabola shown is defined such that all points on the parabola are equidistant from the point (0, 2) and the line $y = -2$. One point on the parabola is labeled as (x, y).



1. Determine the equation of the parabola by completing the steps.

a. Let d_1 represent the distance from (x, y) to (0, 2). Write an equation using the Distance Formula to represent d_1. Simplify the equation.

b. Let d_2 represent the distance from (x, y) to the line $y = -2$. Write an equation using the Distance Formula to represent d_2. Simplify the equation.

c. What do you know about the relationship between d_1 and d_2?

d. Write an equation for the parabola using Question 1, parts (a) through (c). Simplify the equation so that one side of the equation is x^2.

The **general form of a parabola** with a vertex at the origin is an equation of the form $Ax^2 + Dy = 0$ or $By^2 + Cx = 0$.

The **standard form of a parabola** with a vertex at the origin is an equation of the form $x^2 = 4py$ or $y^2 = 4px$, where p represents the distance from the vertex to the focus.

2. **Write the equation of the parabola from Question 1 in general form and in standard form.**

3. **What are the coordinates for the intercept(s) of the parabola?**

4. **Describe the symmetry of the parabola.**

5. **Calculate the coordinates of the point that has an *x*-coordinate of 4.**

6. **Use symmetry to determine which other point on the parabola has a *y*-coordinate of 2. Explain your reasoning.**

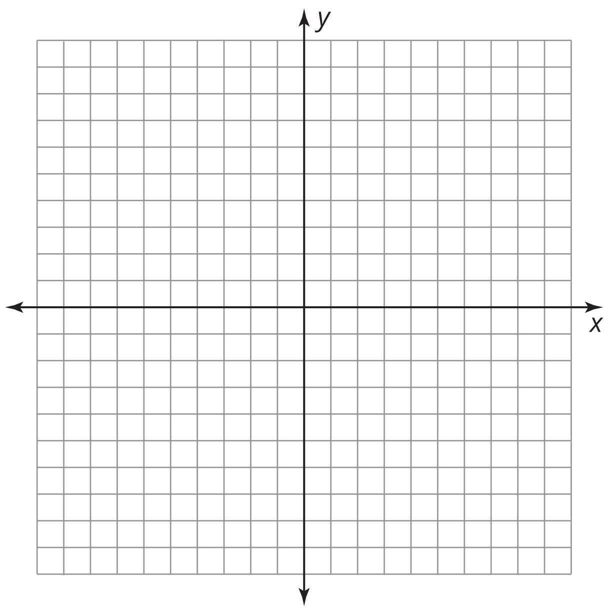
7. **Calculate the coordinates of each point that has a *y*-coordinate of 4.5.**

8. How can you determine the points on the parabola that have a *y*-coordinate of −4.5 graphically and algebraically?

9. Consider the parabola represented by the equation $y^2 = 2x$.

a. Sketch the parabola using the table of values.

x	*y*
8	−4
2	−2
0	0
2	2
8	4



Ask yourself:

What does the form of the equation tell you about the shape of the parabola?

b. What is the relationship between the axis of symmetry and the equation of the parabola?

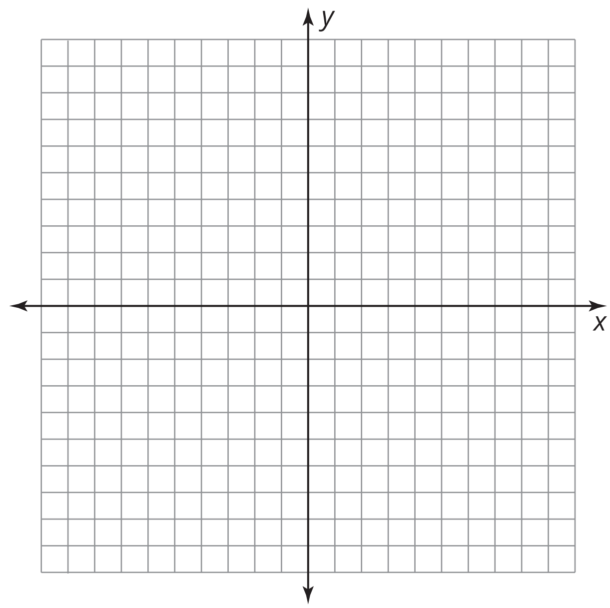
c. What are the coordinates of the vertex?

d. How is the concavity of the parabola related to the orientation of the parabola?

10. **Consider the parabola represented by the equation $x^2 = 9y$.**

   a. **Complete the table of values for the equation. Sketch the parabola using the coordinates from the table.**

x	y
−6	
−3	
0	
3	
6	



   b. **What is the relationship between the axis of symmetry and the equation of the parabola?**

   c. **What are the coordinates of the vertex?**

   d. **How is the concavity of the parabola related to the orientation of the parabola?**

11. **The standard form of a parabola with its vertex at the origin is an equation of the form $x^2 = 4py$ or $y^2 = 4px$.**

   a. **What is the standard form of a parabola with an axis of symmetry along the y-axis?**

b. What is the standard form of a parabola with an axis of symmetry along the *x*-axis?

c. What is the equation of the axis of symmetry for a parabola with a vertical orientation?

d. What is the equation of the axis of symmetry for a parabola with a horizontal orientation?

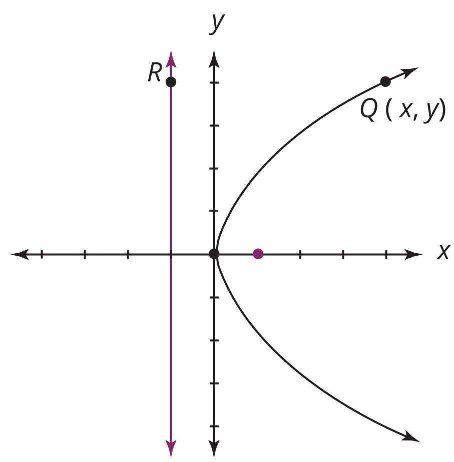
e. Is the concavity of a parabola with a vertical orientation described as concave up/down or concave right/left?

f. Is the concavity of a parabola with a horizontal orientation described as concave up/down or concave right/left?

ACTIVITY 5.3	Making Sense of the Constant p

You have learned that the standard form of a parabola with a vertex at the origin is an equation of the form $x^2 = 4py$ or $y^2 = 4px$, where p represents the distance from the vertex to the focus. What is the significance of the value $4p$ in each equation?

1. Consider the sketch of the parabola. Let *p* represent the distance from the vertex to the focus.



a. Label the vertex with its coordinates.

b. Label the distance, *p*, on the graph.

c. Label the focus with its coordinates.

d. Label the directrix with the equation for its line. Explain your reasoning.

e. What is the distance from the focus to the directrix? Label this distance on the graph.

Think about:

What is the relationship between the directrix and the axis of symmetry?

2. **Write an equation for the parabola.**

   a. **Let d_1 represent the distance from point *Q* on the parabola to the focus. Write an equation using the Distance Formula to represent d_1. Simplify the equation.**

   b. **Line segment *QR* represents the perpendicular distance from point *Q* on the parabola to the directrix. Draw line segment *QR*. What are the coordinates of point *R*? Label the coordinates on the graph.**

   c. **Let d_2 represent the distance from point *Q* to point *R*. Write an equation using the Distance Formula to represent d_2. Simplify the equation.**

   d. **Write an equation for the parabola using parts (a) through (c). Simplify the equation so that one side of the equation is the squared term.**

   e. **Describe the significance of the equation derived in part (d).**

ACTIVITY **5.4**

Using the Constant *p* to Graph a Parabola

Let's investigate different equations to understand the relationships between the structure of the equation and its graph.

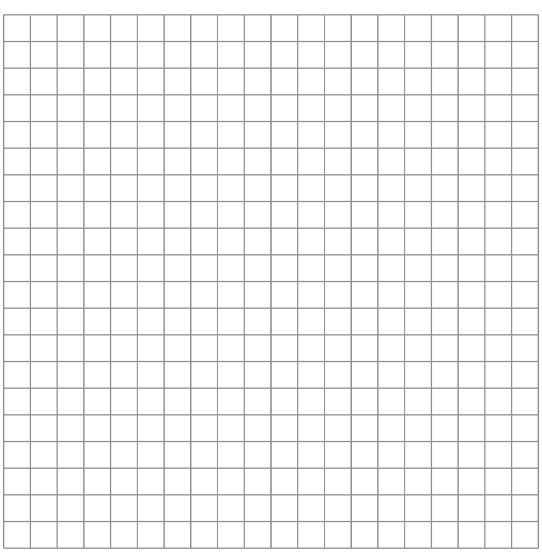
Ask yourself:

What does the form of the equation tell you? What do you think the graph will look like?

1. Consider the parabola represented by the equation $y^2 = 20x$.

a. Identify the coordinates of the vertex and the equation of the line of symmetry.

b. Determine the value of *p*. Then determine the coordinates of the focus and the equation of the directrix. Justify your reasoning.



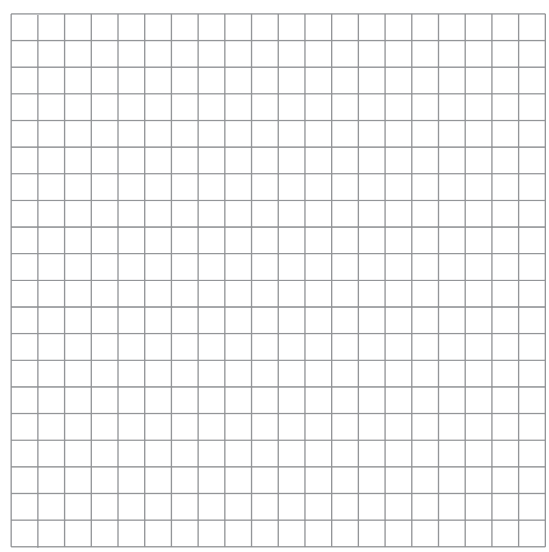
c. Graph the parabola. Then describe the concavity of the parabola. Justify your reasoning.

2. Consider the parabola represented by the equation $x^2 = -12y$.

a. Identify the coordinates of the vertex and the equation of the line of symmetry.

b. Determine the value of p. Then determine the coordinates of the focus and the equation of the directrix. Justify your reasoning.

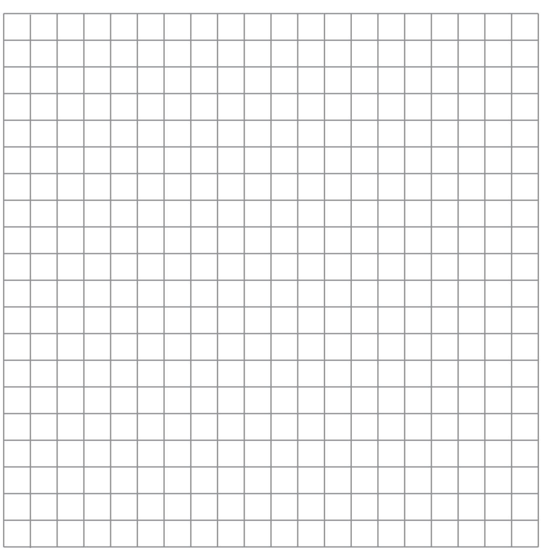
c. Graph the parabola. Then describe the concavity of the parabola. Justify your reasoning.



3. **Consider the parabola represented by the equation $x^2 = 28y$.**

   a. **Identify the coordinates of the vertex and the equation of the line of symmetry.**

   b. **Determine the value of p. Then determine the coordinates of the focus and the equation of the directrix. Justify your reasoning.**



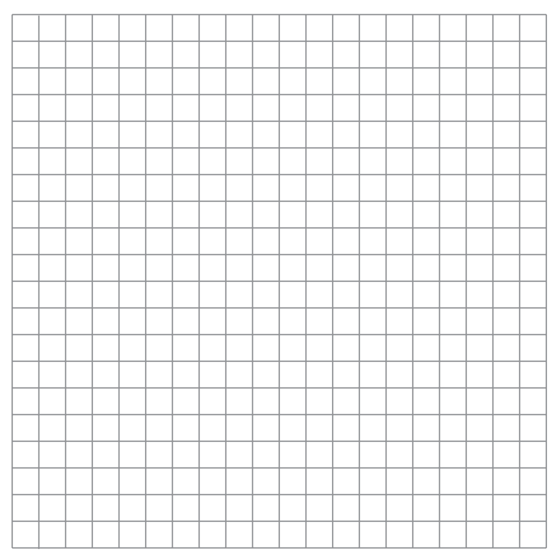
   c. **Graph the parabola. Then describe the concavity of the parabola. Justify your reasoning.**

4. Consider the parabola represented by the equation $y^2 = -10x$.

a. Identify the coordinates of the vertex and the equation of the line of symmetry.

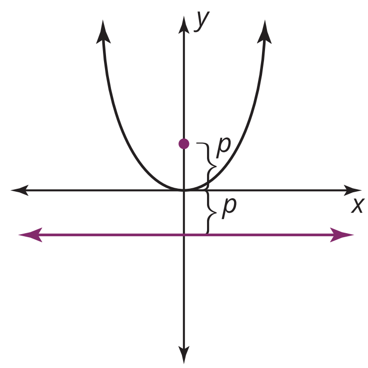
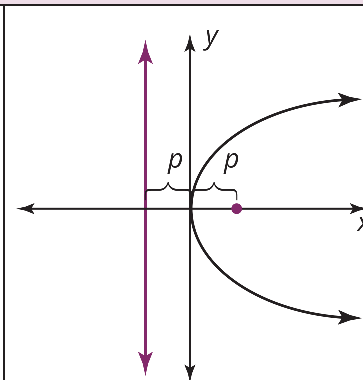
b. Determine the value of *p*. Then determine the coordinates of the focus and the equation of the directrix. Justify your reasoning.

c. Graph the parabola. Then describe the concavity of the parabola. Justify your reasoning.



5. **Analyze each equation and its corresponding graph in Questions 1 through 4. Describe the relationship between the sign of the constant *p* and the concavity of each parabola.**

6. **Complete the table.**

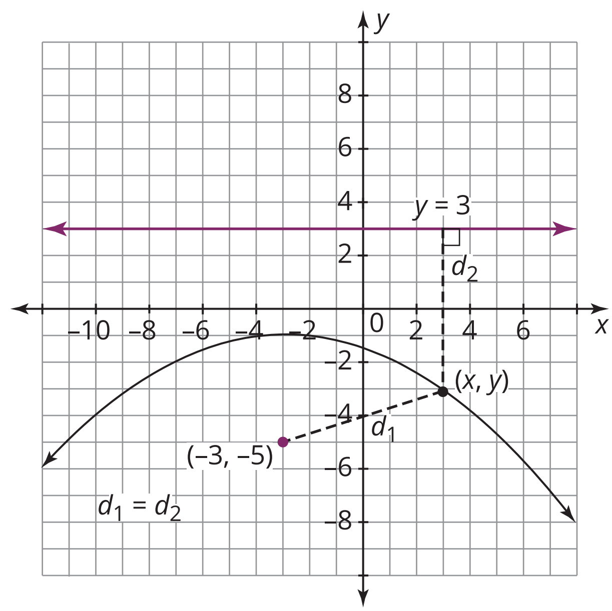
Parabola Centered at Origin		
Graph		
Equation of Parabola		
Orientation of Parabola		
Axis of Symmetry		
Coordinates of Vertex		
Coordinates of Focus		
Equation of Directrix		
Concavity		

ACTIVITY 5.5 Writing an Equation Given a Focus and a Directrix

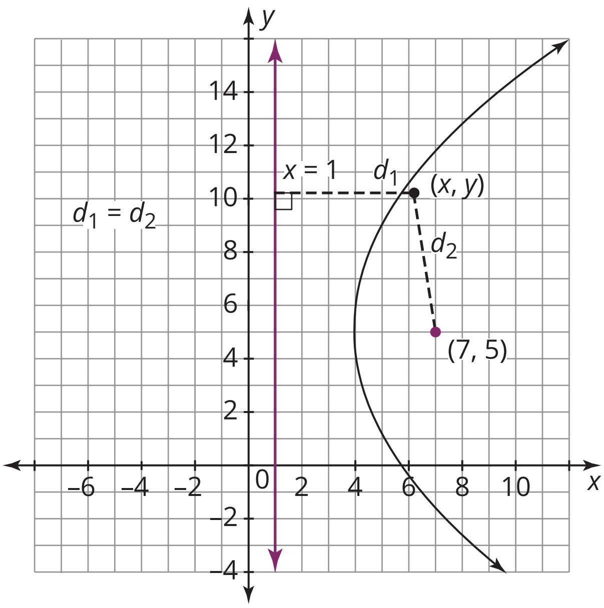
In this activity, you will use the Distance Formula to determine the equation of points that are equidistant from a given point (the focus) and a given line (the directrix) where the vertex is a point other than the origin.

1. Consider the graph shown.

a. Determine an equation for all the points equidistant from the point (−3, −5) and the line $y = 3$.



b. Determine the coordinates of the vertex and equation of the axis of symmetry of the parabola. Explain your reasoning.



2. **Consider the graph shown.**

   a. **Determine the equation for all the points equidistant from the point (7, 5) and the line $x = 1$.**

   b. **Determine the coordinates of the vertex and equation of the axis of symmetry of the parabola. Explain your reasoning.**

The standard forms of parabolas with vertex at (h, k) are $(x - h)^2 = 4p(y - k)$ and $(y - k)^2 = 4p(x - h)$.

3. **Rewrite the equations from Questions 1 and 2 in one of these forms.**

Think about:

How do these equations compare with the standard form of the equation of a circle: $(x - h)^2 + (y - k)^2 = r^2$?

4. **Rewrite each equation in standard form. Determine the value of *p*, the coordinates of the vertex and focus, and the equations of the directrix and the axis of symmetry. Then sketch the graph of the parabola with the focus and directrix and describe the concavity.**

   a. $y^2 + 8y + 8x + 16 = 0$

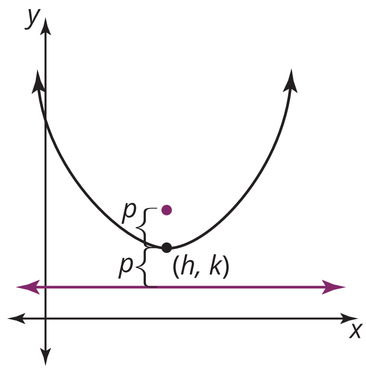
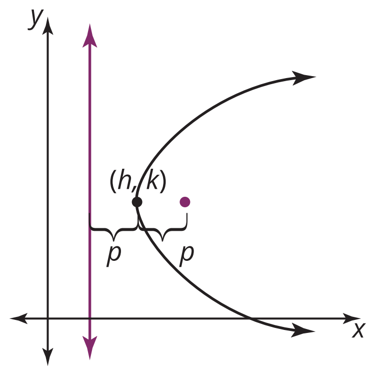
   b. $4x^2 - 40x + 48y + 4 = 0$

5. **Write an equation in standard form for each parabola. Then, graph and label the parabola.**

a. **A parabola with a vertex at (3, 2) and a focus at (3, 4).**

b. **A parabola with a vertex at (4, 1) and a directrix at $x = 2$.**

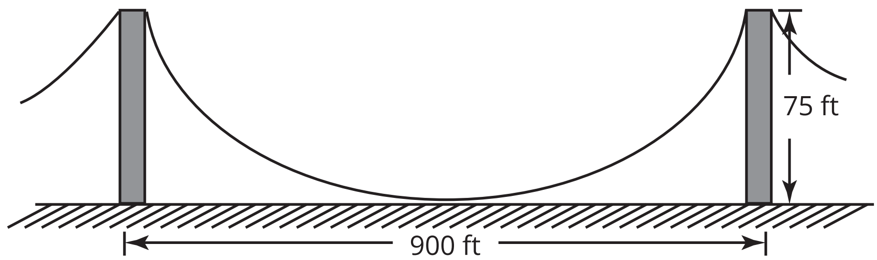
6. Complete the table.

Parabola		
Graph		
Equation of Parabola	$(x - h)^2 = 4p(y - k)$	$(y - k)^2 = 4p(x - h)$
Orientation of Parabola		
Axis of Symmetry		
Coordinates of Vertex		
Coordinates of Focus		
Equation of Directrix		
Concavity		

ACTIVITY
5.6

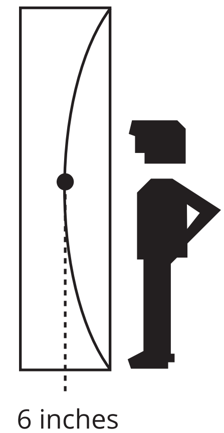
Applications of Parabolas

1. The main cables of a suspension bridge are parabolic. The parabolic shape allows the cables to bear the weight of the bridge evenly. The distance between the towers is 900 feet and the height of each tower is about 75 feet. Write an equation for the parabola that represents the cable between the two towers.



2. A cross-section of a satellite dish is a parabola. The satellite dish is 5 feet wide at its opening and 1 foot deep. The receiver of the satellite dish should be placed at the focus of the parabola. How far should the receiver be placed from the vertex of the satellite dish?

3. Many carnivals and amusement parks have mirrors that are parabolic. When you look at your reflection in a parabolic mirror, your image appears distorted and makes you look taller or shorter depending on the shape of the mirror. The focal length of a mirror is the distance from the vertex to the focus of the mirror. Consider a mirror that is 72 inches tall with a vertex that is 6 inches from the top and bottom edges of the mirror. What is the focal length of the mirror?

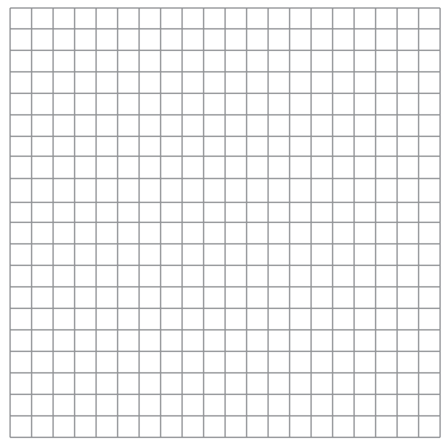


TALK the TALK

Simply Parabolic

Graph each parabola. Label the vertex, the focus, and the directrix. Then describe the concavity.

1. $x^2 = 18y$



2. $y^2 + 44x = 0$