5

Going the Equidistance

Equation of a Parabola

MATERIALS

Compasses
Patty paper
Straightedges

Lesson Overview

The focus and directrix of a parabola are introduced through an exploratory activity. Students use concentric circles to plot points that are equidistant from both a line and a point not on the line, then connect these equidistant points to form a parabola. A parabola is described as a conic section, and the terms *locus of points, parabola, focus,* and *directrix* are given. Students construct a directrix and a focus above the directrix on patty paper and complete multiple folds of the focus onto the line to create a parabola. *Concavity* and *vertex of a parabola* are defined. Through investigations, students conclude that any point on a parabola is equidistant from the focus and the directrix. The focus and directrix are then used to write the equation of a parabola, and the general and standard form of a parabola are given. Students derive the standard form of a parabola algebraically to make sense of the constant *p* in the equation and use this constant to graph parabolas. The Distance Formula is used to determine the equation of points that are equidistant from a given focus and a given directrix where the vertex is a point other than the origin. Students apply characteristics of parabolas to solve real-world problem situations.

Algebra 2
Quadratic and Square Root Functions, Equations, and Inequalities

(4) The student applies mathematical processes to understand that quadratic and square root functions, equations, and quadratic inequalities can be used to model situations, solve problems, and make predictions. The student is expected to:

(B) write the equation of a parabola using given attributes, including vertex, focus, directrix, axis of symmetry, and direction of opening.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 5.B, 5.F, 5.G

Essential Ideas

- A parabola is the locus of points in a plane that are equidistant from a fixed point (the focus) and a fixed line (the directrix).
- The focus and directrix of a parabola can be used to derive the equation of the parabola.

- Parabolas can be described by their concavity.
- The standard form for the equation of a parabola with vertex at the origin can be written in the form $x^2 = 4py$ (symmetric with respect to the y-axis) or $y^2 = 4px$ (symmetric with respect to the x-axis), where p is the distance from the vertex to the focus.
- The standard form for the equation of a parabola with vertex at the origin, $y^2 = 4px$ or $x^2 = 4py$ can be derived using the Distance Formula and the definitions of focus, directrix, and parabola.
- In the standard form for the equation of a parabola centered at the origin, $y^2 = 4px$ or $x^2 = 4py$ the value of p is positive when the parabola is concave up or concave right and the value of p is negative when the parabola is concave down or concave left.
- The standard forms of parabolas with vertex (h, k) are $(x - h)^2 = 4p(y - k)$ and $(y - k)^2 = 4p(x - h)$.
- The characteristics of parabolas can be used to solve real-world problems.

Lesson Structure and Pacing: 3 Days

Day 1

Engage

Getting Started: Hocus Pocus . . . Make the Locus!

Students use concentric circles to plot points that are equidistant from both a line and a point not on the line, then connect these equidistant points to form a parabola.

Develop

Activity 5.1: A Parabola as a Locus of Points

A parabola is described as a conic section. The terms *locus of points, parabola*, *focus*, and *directrix* are given. Students construct a directrix and a focus above the directrix on patty paper and complete multiple folds of the focus onto the line to create a parabola. *Concavity* and *vertex of a parabola* are defined. Through investigations students conclude that any point on a parabola is equidistant from the focus and the directrix.

Activity 5.2: Equations of a Parabola

Students determine the equation of a parabola using the Distance Formula, a fixed point, and a fixed line. The general form of a parabola centered at the origin, $Ax^2 + Cy = 0$ or $By^2 + Cx = 0$, is given. Students are also given the standard form of a parabola centered at the origin, $x^2 = 4py$ or $y^2 = 4px$, where p is the distance from the vertex to the focus. Connections are established between the key characteristics of a parabola and the positions in which they appear when the equation for a parabola is written in standard form.

Day 2

Activity 5.3: Making Sense of the Constant *p*

Students use the distance p from the vertex to the focus and the Distance Formula to algebraically derive the equation of a parabola written in the form $y^2 = 4px$. Deriving the equation illustrates the significance of $4p$.

Activity 5.4: Using the Constant *p* to Graph a Parabola

Given the equations of four parabolas, students identify the coordinates of the vertex, the equation of the axis of symmetry, the focus, and the equation of the directrix. After graphing each of the parabolas on a coordinate plane, students note the relationship between the p-value and characteristics of the graphical representation. Connections are made between concavity and the location of the focus and the directrix.

Day 3

Activity 5.5: Writing an Equation Given a Focus and a Directrix

Students determine equations of parabolas with vertices not at the origin given the focus and directrix, then identify key characteristics of the graph. They are given the standard form of a parabola with vertex at (h, k): $(x - h)^2 = 4p(y - k)$ or $(y - k)^2 = 4p(x - h)$. Students rewrite the equation of a parabola in standard form, identify its key characteristics, and sketch its graph. They also write an equation in standard form given either the vertex and the focus or the vertex and the directrix, then sketch the graph. Students complete a table that summarizes the key characteristics of parabolas with different orientations and vertices not at the origin.

Activity 5.6: Applications of Parabolas

Students use the characteristics of parabolas to solve real-world problems.

Demonstrate

Talk the Talk: Simply Parabolic

Students graph parabolas given equations. They then label the parabola with its key characteristics: the vertex, focus, and directrix. Students describe the concavity of each.

ENGAGE

Getting Started: Hocus Pocus . . . Make the Locus!

Facilitation Notes

In this activity, students use concentric circles to plot points that are equidistant from both a line and a point not on the line, then connect these equidistant points to form a parabola.

Have students work with a partner or in a group to complete Questions 1 through 4. Share responses as a class.

Differentiation strategy

To scaffold support, instruct students to draw all of the concentric circles first, then suggest they use colored pencils to color code each of the locations where a horizontal distance line intersects a corresponding circle that has radii of equal distance to the line, using a different color for each line and pair of points. Students could locate the points on one side of the parabola, then use symmetry to locate the points on the second side of the parabola.

As students work, look for

The smooth curve created by the equidistant points that is concave upward. There should be two points located on each concentric circle with the exception of the 1 and 2 unit circles.

Misconception

Students may not fully understand how the smooth parabolic curve is a result of equidistance. They have studied equidistance that resulted in the formation of a perpendicular bisector and equidistance related to the locus of points that form a circle. Understanding a *single* point that is equal in distance from both a fixed point and a fixed line is easy to visualize, but locating a locus of points that are equal in distance requires students to think of points that are not oriented in between the given point and fixed line.

Questions to ask

- Are all of the points on the circumference of the circle equidistant from the center point?
- If the radius of the circle is one unit, are all points on the circumference of the circle one unit from the center point?
- Where are all of the points that are one unit from the fixed line?
- Are all of the points that are one unit from the fixed line located on a horizontal line? Where is that horizontal line?
- What is the distance between each concentric ring?
- Where is the single point that is two units from the center point and two units from the fixed line?

- Where are the two points that are three units from the center point and three units from the fixed line?
- Where are the two points that are four units from the center point and fours units from the fixed line?
- How did you locate the additional points that satisfy the criteria?
- Why don't any points that satisfy the criteria appear below the fixed line?
- Is the shape of the smooth curve parabolic?
- Are all of the points equidistant from the center point and the fixed line?

Summary

A smooth curve in the shape of a parabola is formed by plotting and connecting points that are equidistant from a given point and line on a coordinate grid.

Activity 5.1

A Parabola as a Locus of Points

DEVELOP

Facilitation Notes

In this activity, a parabola is described as a conic section. The terms *locus of points*, *parabola*, *focus*, and *directrix* are given. Students construct a directrix and a focus above the directrix on patty paper and complete multiple folds of the focus onto the line to create a parabola. *Concavity* and *vertex of a parabola* are defined. Through investigations, students conclude that any point on a parabola is equal distance from the focus and the directrix.

Ask a student to read the introduction aloud and discuss as a class. Have students work with a partner or in a group to complete Question 1. Share responses as a class.

As students work, look for

Alignment of point *F* on the line each time before they make a crease.

Differentiation strategy

To scaffold support, have students mark 30–40 points on the line before they begin the folding process. The more points on the line, the more defined the parabola will look after the folds and creases have been made.

Questions to ask

- What type of symmetry does your parabola have?
- How would the shape of the parabola change if you had drawn point *F* closer to the line? Farther away from the line?
- What would have happened if you had drawn point *F* on the line?

Ask a student to read the paragraph after Question 1 aloud and discuss as a class. Have students work with a partner or in a group to complete Questions 2 through 4. Share responses as a class.

As students work, look for

Students drawing point *D* on the directrix.

Questions to ask

- What type of concavity does your parabola have?
- How do you know that $\overline{PF}$ and $\overline{PD}$ are congruent?
- What did you discover when you drew other points on the directrix?

Summary

Given a directrix and a focus above the directrix, you can construct a parabola by repeatedly folding the focus onto the directrix and making creases. Any point on a parabola is equidistant from the focus and the directrix.

Activity 5.2

Equations of a Parabola

Facilitation Notes

In this activity, students determine the equation of a parabola using the Distance Formula, a fixed point, and a fixed line. The general form of a parabola centered at the origin, $Ax^2 + Cy = 0$ or $By^2 + Cx = 0$, is given. Students are also given the standard form of a parabola centered at the origin, $x^2 = 4py$ or $y^2 = 4px$, where p is the distance from the vertex to the focus. Connections are established between the key characteristics of a parabola and the positions in which they appear when the equation for a parabola is written in standard form.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

As students work, look for

An incorrect assumption. Using the graph, students may assume that $x = 5$ because the point at which the perpendicular segment d_2 intersects the horizontal line $y = -2$ is close to the horizontal distance of 5 on the x-axis. The coordinates of this point of intersection is $(x, -2)$, not $(5, -2)$.

Questions to ask

- In this situation, where is the focus? Where is the directrix?
- Is the point (x, y) closer to the focus or the directrix?
- When solving for the value of d_1, the coordinates of which two points were substituted into the Distance Formula?

- When solving for the value of d_2, the coordinates of which two points were substituted into the Distance Formula?
- Why is $d_1 = d_2$?
- What is the simplification of the expression $\sqrt{(x - 0)^2 + (y - 2)^2}$?
- What is the simplification of the expression $\sqrt{(x - x)^2 + (y + 2)^2}$?
- How can the expression $(y - 2)^2$ be written as a trinomial?
- How can the expression $(y + 2)^2$ be written as a trinomial?
- Which like terms appear on both sides of the equation? Can they be subtracted from both sides to simplify the equation?
- Is the equation for the parabola, the quadratic equation $x^2 = 8y$?

Ask a student to read the definitions following Question 1 aloud. Discuss as a class.

Have students work with a partner or in a group to complete Questions 2 through 8. Share responses as a class.

Differentiation strategy

To assist all students, suggest that they represent the equation of the parabola, $x^2 = 8y$, using a table of values before they answer the questions.

Questions to ask

- What is the difference between the general form of the equation for a parabola and the standard form of the equation for a parabola?
- Given the equation of a parabola, how do you determine the x-intercepts? The y-intercepts?
- For every x-value, is there one or more than one y-value on a parabola? How do you know?
- For every y-value, is there one or more than one x-value on a parabola? How do you know?
- What about the equation $x^2 = 8y$ suggests the vertical orientation of the parabola?
- What about the equation $x^2 = 8y$ suggests if it opens up or down?
- If $y = 2$ in the equation $x^2 = 8y$, is the value of x equal to both 4 and -4?
- If $y = 4.5$ in the equation $x^2 = 8y$, is the value of x equal to both 6 and -6?
- If the radicand is negative, what does this imply about the algebraic solution? What does this imply about the graph of the equation?

Have students work with a partner or in a group to complete Questions 9 and 10. Share responses as a class.

Questions to ask

- For every x-value, is there one or more than one y-value on a parabola? How do you know?

- For every y-value, is there one or more than one x-value on a parabola? How do you know?
- What about the equation $y^2 = 2x$ suggests the horizontal orientation of the parabola?
- What about the equation $y^2 = 2x$ suggests if it opens to the left or to the right?
- Are the coordinates of the vertex included in the table of values?
- If $x = -6$ in the equation $x^2 = 9y$, how did you determine the value of y?
- When $x = \pm 6$ in the equation $x^2 = 9y$, what is the value of y?
- How does the graph of $x^2 = 9y$ compare to the graph of $x^2 = 8y$?
- Without graphing, how would you describe the graph of $x^2 = 7y$?

Have students work with a partner or in a group to complete Question 11. Share responses as a class.

Questions to ask

- If the axis of symmetry is along the x-axis, what is the orientation of the parabola?
- If the axis of symmetry is along the x-axis do two different points on the parabola share the same x-value or do they share the same y-value?
- If the axis of symmetry is along the y-axis, what is the orientation of the parabola?
- If the axis of symmetry is along the y-axis, do two different points on the parabola share the same x-value or do they share the same y-value?
- Why does a parabola written in the form $x^2 = 4py$ open up or down?
- Why does a parabola written in the form $y^2 = 4px$ open to the left or to the right?
- Is $x = 0$ the x-axis or the y-axis? Is $y = 0$ the x-axis or the y-axis?
- Why is $x = 0$ the y-axis? Why is $y = 0$ the x-axis?

Differentiation strategy

To extend the activity, have students write each of the general forms of a parabola and each of the standard forms of a parabola on poster paper. Include an example, a graph, and a table of values. Display the posters so students can associate the equations with their orientations as they continue to study this topic.

Summary

The general form of a parabola centered at the origin is an equation of the form $Ax^2 + Dy = 0$ or $By^2 + Cx = 0$. The standard form of a parabola centered at the origin is an equation of the form $x^2 = 4py$ or $y^2 = 4px$.

Activity 5.3
Making Sense of the Constant *p*

Facilitation Notes

In this activity, students use the distance p from the vertex to the focus and the Distance Formula to algebraically derive the equation of a parabola written in the form $y^2 = 4px$. Deriving the equation illustrates the significance of $4p$.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

As students work, look for

The application of prior knowledge. If students do not understand the definitions of *parabola*, *directrix*, *vertex*, and *focus*, choosing the right coordinates to represent the points in this derivation is more difficult. Review the definitions as needed and discuss their meaning in terms of the graph.

Questions to ask

- In this situation, is the vertex of the parabola located at the origin?
- What are the coordinates of the origin?
- What is the distance from the vertex to the focus?
- What is the distance from the vertex to the directrix?
- How would you describe the orientation of the parabola?
- What given information helped you determine the coordinates of the focus?
- What do you know about the location of a directrix, with respect to the location of the focus?
- Why is the equation for the directrix $x = -p$?
- What are the coordinates of a point that lies on both the line of symmetry and the directrix?
- If the distance from the focus to the vertex is p and the distance from the vertex to the directrix is p, what expression represents the distance from the focus to the directrix?

Have students work with a partner or in a group to complete Question 2. Share responses as a class.

Questions to ask

- Which points were used to determine the value of d_1?
- Which points were used to determine the value of d_2?
- Why are the coordinates of point R $(-p, y)$?
- Why is $d_1 = d_2$?
- Is the equation for the parabola in the form of $x^2 =$, or $y^2 =$?

- If an equation of a parabola is written in standard form, what information about the parabola can easily be determined?

Differentiation strategy

To extend the activity, have students derive $x^2 = 4py$. Begin by sketching a parabola oriented vertically upward, label the focus coordinates $(0, p)$, the coordinates of the point on the parabola (x, y), the equation of the directrix $y = -p$, and the coordinates of the point on the directrix $(x, -p)$.

Summary

The standard form for the equation of a parabola centered at the origin, $y^2 = 4px$ or $x^2 = 4py$ can be derived using the Distance Formula and the definitions of *focus, directrix,* and *parabola*.

Activity 5.4

Using the Constant *p* to Graph a Parabola

Facilitation Notes

In this activity, given the equations of four parabolas, students identify the coordinates of the vertex, the equation of the axis of symmetry, the focus, and the equation of the directrix. After graphing each of the parabolas on a coordinate plane, students note the relationship between the *p*-value and characteristics of the graphical representation. Connections are made between concavity and the location of the focus and the location of the directrix.

Differentiation strategy

As an alternate grouping strategy, assign different sections of the class only one of the four questions. Provide time for students to share their solutions with the class. Each question describes a parabola that has a different concavity. Then answer Question 5 as a class.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Differentiation strategy

To scaffold support, suggest that students create a table of values for each of the equations before answering the questions.

Questions to ask

- In the equation $y^2 = 20x$, when $x = 0$, what is the value of y?
- For every *x*-value, is there one or more than one *y*-value on a parabola? How do you know?

- For every y-value, is there one or more than one x-value on a parabola? How do you know?
- If there are two y-values for most values of x, does this suggest a horizontal orientation or a vertical orientation?
- If the graph of the equation of the parabola has a horizontal orientation, does it follow that the directrix will be a vertical line or a horizontal line?
- What about the equation $y^2 = 20x$ suggests it opens to the left or to the right?
- Why does $20x = 4px$?
- If the value of $p = 5$, what implications does this have on the graph of the parabola?
- How is the value of p used to determine the coordinates of the focus?
- How is the value of p used to write the equation of the directrix?
- Does the parabola open toward the focus or away from the focus?
- Does the parabola open toward the directrix or away from the directrix?
- In the equation $x^2 = -12y$, when $y = 0$, what is the value of x?
- For every x-value, is there one or more than one y-value on a parabola? How do you know?
- For every y-value, is there one or more than one x-value on a parabola? How do you know?
- If there are two x-values for most values of y, does this suggest a horizontal orientation or a vertical orientation?
- If the graph of the equation of the parabola has a vertical orientation, does it follow that the directrix will be a vertical line or a horizontal line?
- What about the equation $x^2 = -12y$ suggests it opens up or down?
- Why does $-12y = 4py$?
- If the value of $p = -3$, what implications does this have on the graph of the parabola?
- How is the value of p used to determine the coordinates of the focus?
- How is the value of p used to write the equation of the directrix?
- Does the parabola open toward the focus or away from the focus?
- Does the parabola open toward the directrix or away from the directrix?

Have students work with a partner or in a group to complete Questions 3 and 4. Share responses as a class.

Questions to ask

- In the equation $x^2 = 28y$, when $y = 0$, what is the value of x?
- For every x-value, is there one or more than one y-value on a parabola? How do you know?
- For every y-value, is there one or more than one x-value on a parabola? How do you know?
- If there are two x-values for most values of y, does this suggest a horizontal orientation or a vertical orientation?
- If the graph of the equation of the parabola has a vertical orientation, does it follow that the directrix is a vertical line or a horizontal line?
- What about the equation $x^2 = 28y$ suggests it opens up or down?
- Why does $28y = 4py$?
- If the value of $p = 7$, what implications does this have on the graph of the parabola?
- How is the value of p used to determine the coordinates of the focus?
- How is the value of p used to write the equation of the directrix?
- Does the parabola open toward the focus or away from the focus?
- Does the parabola open toward the directrix or away from the directrix?
- In the equation $y^2 = -10x$, when $x = 0$, what is the value of y?
- For every x-value, is there one or more than one y-value on a parabola? How do you know?
- For every y-value, is there one or more than one x-value on a parabola? How do you know?
- If there are two y-values for most values of x, does this suggest a horizontal orientation or a vertical orientation?
- If the graph of the equation of the parabola has a horizontal orientation, does it follow that the directrix will be a vertical line or a horizontal line?
- What about the equation $y^2 = -10x$ suggests it opens to the left or to the right?
- Why does $-10x = 4px$?
- If the value of $p = -2.5$, what implications does this have on the graph of the parabola?
- How is the value of p used to determine the coordinates of the focus?
- How is the value of p used to write the equation of the directrix?
- Does the parabola open toward the focus or away from the focus?
- Does the parabola open toward the directrix or away from the directrix?

Have students work with a partner or in a group to complete Question 5. Share responses as a class.

Questions to ask

- What do the p-values in the equations $y^2 = 20x$ and $x^2 = 28y$ have in common?
- What do the p-values in the equations $x^2 = -12y$ and $y^2 = -10x$ have in common?
- Which concavities are associated with positive p-values?
- Which concavities are associated with negative p-values?
- When $p = 5$ or $p = 7$, is the concavity up or to the right?
- When $p = -3$ or $p = -2.5$, is the concavity down or to the left?

Have students work with a partner or in a group to complete Question 6. Share responses as a class.

As students work, look for

The necessary connections made to identify various characteristics of each parabola. Discuss what minimal information must be given to reveal all of the characteristics listed in the table in Question 6. Review each characteristic, one at a time, and ask students, "If __________ is known, what other characteristics can be identified?"

Questions to ask

- What aspect of the equation indicates the location of the vertex?
- What aspect of the equation indicates a vertical or horizontal orientation?
- How is $x^2 = 4py$ used to determine the p-value?
- Once the p-value is known, what additional characteristics of the parabola can be revealed?
- How are the coordinates of the vertex used to identify the axis of symmetry?
- What aspect of the equation indicates the concavity?

Differentiation strategy

To extend the activity, have students expand the table in Question 6 by completing two additional columns. Ask them to draw the parabola that opens down and the parabola that opens to the left.

Summary

In the standard form for the equation of a parabola centered at the origin, $y^2 = 4px$ or $x^2 = 4py$, the value of p is positive when the parabola is concave up or concave right and the value of p is negative when the parabola is concave down or concave left.

Activity 5.5
Writing an Equation Given a Focus and a Directrix

Facilitation Notes

In this activity, students determine equations of parabolas with a vertices not at the origin given the focus and directrix, then identify key characteristics of the graph. They are given the standard form of a parabola with vertex at (h, k): $(x - h)^2 = 4p(y - k)$ or $(y - k)^2 = 4p(x - h)$. Students rewrite the equation of a parabola in standard form, identify its key characteristics, and sketch its graph. They also write an equation in standard form given either the vertex and the focus or the vertex and the directrix, then sketch the graph. Students complete a table that summarizes the key characteristics of parabolas with different orientations and vertices not at the origin.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

Questions to ask

- If all of the points are equidistant from point $(-3, -5)$ and the line $y = 3$, do the points form a parabola? is the point located at $(-3, -5)$ the focus of the parabola? Is $y = 3$ the directrix?
- When solving for the value of d_1, the coordinates of which two points were substituted into the Distance Formula?
- When solving for the value of d_2, the coordinates of which two points were substituted into the Distance Formula?
- Why is $d_1 = d_2$?
- How do you rewrite the expression $\sqrt{(x + 3)^2 + (y + 5)^2}$?
- How do you rewrite the expression $\sqrt{(x - x)^2 + (y - 3)^2}$?
- How can the square of a binomial, $(a + b)^2$, be written as a trinomial? The binomial $(a - b)^2$?
- Which like terms appear on both sides of the equation? Can they be subtracted from both sides to simplify the equation?
- How is this equation for the parabola different than the equations you previously wrote?
- Is the axis of symmetry a vertical line through the x-coordinate or the y-coordinate of the focus?
- Is the vertex 4 units away from both the focus and the directrix?
- If all of the points are equidistant from point $(7, 5)$ and the line $x = 1$, do the points form a parabola? is the point located at $(7, 5)$ the focus of the parabola? Is $x = 1$ the directrix?
- When solving for the value of d_1, the coordinates of which two points were substituted into the Distance Formula?

- When solving for the value of d_2, the coordinates of which two points were substituted into the Distance Formula?
- How do you rewrite the expression $\sqrt{(x - 1)^2 + (y - y)^2}$?
- How do you rewrite the expression $\sqrt{(x - 7)^2 + (y - 5)^2}$?
- Which like terms appear on both sides of the equation? Can they be subtracted from both sides to simplify the equation?
- How is this equation for the parabola different than the equation in Question 1?
- Is the vertex 3 units away from both the focus and the directrix?
- Do you need to complete the square to write $x^2 + 6x + 16y + 25 = 0$ in the form $(x - h)^2 = 4p(y - k)$?
- What are the coordinates of the point (h, k)? What is the p-value?
- Do you need to complete the square to write $y^2 - 10y - 12x + 73 = 0$ in the form $(y - k)^2 = 4p(x - h)$?
- What are the coordinates of the point (h, k)? What is the p-value?

Have students work with a partner or in a group to complete Questions 4 and 5. Share responses as a class.

As students work, look for

The steps they used to transform the equation into standard form. Did they begin by moving the constant term, 16, to the right side of the equation, then complete the square on the left side and end up adding 16 to both sides? If so, they duplicated their efforts. Point out that they didn't have to move the constant term 16, they could have just moved the term, $8x$, to the right side and factored the left side of the equation $y^2 + 8y + 16$ because it was the perfect square $(y + 4)^2$.

Questions to ask

- If the equation contains a y^2 term, what implications does this have on the orientation of the parabola?
- Is the axis of symmetry horizontal or vertical?
- If the equation of the parabola is $(y + 4)^2 = -8x$, what is the p-value? What are the coordinates of the vertex point (h, k)?
- If the vertex is (0, −4) and the focus is (−2, −4), what is the concavity of the parabola?
- If the equation contains a x^2 term, what implications does this have on the orientation of the parabola?
- Is the axis of symmetry horizontal or vertical?
- If the equation of the parabola is $(x - 5)^2 = -12(y - 2)$, what is the p-value? What are the coordinates of the vertex point (h, k)?
- If the vertex is (5, 2) and the focus is (5, −1), what is the concavity of the parabola?

- If the vertex and the focus have the same x-value but different y-values, what implications does this have on the graph of the parabola?
- If the coordinates of the vertex and the focus are known, how do you determine the p-value? The directrix?
- What is the distance between the focus and the vertex?
- If the distance between the focus and vertex is 2 units, what is the distance between the vertex and the directrix?
- Does the equation $(x - h)^2 = 4p(y - k)$ or the equation $(y - k)^2 = 4p(x - h)$ fit this situation?

Have students work with a partner or in a group to complete Question 6. Share responses as a class.

As students work, look for

Use of a common language. Students should all be using the same variables: (h, k) to describe the vertex, the p-value to describe the distance from the focus to the vertex and distance from the vertex to the directrix.

Questions to ask

- Is the orientation of the parabola vertical or horizontal?
- Is the axis of symmetry $x = h$, or $y = k$?
- Are the coordinates of the vertex always represented using (h, k)?
- To determine the coordinates of the focus point, when is the p-value added onto the k-value of (h, k)?
- To determine the coordinates of the focus point, when is the p-value added onto the h-value of (h, k)?
- When is the equation of the directrix written in the form $y = k - p$?
- When is the equation of the directrix written in the form $x = h - p$?
- What implications does a positive p-value have related to the concavity?
- What implications does a negative p-value have related to the concavity?

Differentiation strategy

To extend the activity, have students expand the table in Question 6 by completing two additional columns. Ask them to draw the parabola that opens down and the parabola that opens to the left.

Summary

The standard forms of parabolas with vertex (h, k) are $(x - h)^2 = 4p(y - k)$ and $(y - k)^2 = 4p(x - h)$.

Activity 5.6
Applications of Parabolas

Facilitation Notes

In this activity, students use the characteristics of parabolas to solve real-world problems.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Differentiation strategy

To assist all students with Question 1, suggest they copy the parabolic cable onto patty paper first, then position it over a coordinate plane so they can easily locate the vertex at the origin and identify the coordinates of the points on the top of the towers.

As students work, look for

The equation used to represent the parabolas in Questions 1 and 2. Did students use $x^2 = 4py$ or did they use $(x - h)^2 = 4p(y - k)$? Can either equation be used to solve this problem situation? Is there an advantage to using one equation rather than the other?

Questions to ask

- How can you locate the bridge on a coordinate plane to provide coordinates that you can use to solve the problem?
- If you locate the lowest point of the parabolic cable at the origin, what are the coordinates of the two points located at the top of each tower?
- Why are the coordinates of the two points on the top of the towers (−450, 75) and (450, 75)?
- What is the orientation of the parabola?
- Which standard equation, $y^2 = 4px$ or $x^2 = 4py$, is used to solve for the p-value?
- Why is the equation $x^2 = 4py$ appropriate in this situation?
- How is the p-value of 675 used to write the equation of the parabola?
- How can you locate the satellite dish on a coordinate plane to provide coordinates that you can use to solve the problem?
- If you locate the lowest point of the parabolic dish at the origin, what are the coordinates of two points located at the top left and top right at its opening?
- Using a satellite dish that has a width of 5′ and a depth of 1′, what are the coordinates of two symmetric points located at the top of the rim?
- Why are the coordinates of the two points on the top left and top right of the opening (−2.5, 1) and (2.5, 1)?
- What is the orientation of the parabola?
- Which standard equation, $y^2 = 4px$ or $x^2 = 4py$, is used to solve for the p-value?
- Why is the equation $x^2 = 4py$ appropriate in this situation?

- What point did you use to solve for the p-value?
- Is the p-value the solution to this problem situation?

Have students work with a partner or in a group to complete Question 3. Share responses as a class.

Differentiation strategy

To assist all students with Question 3, suggest they copy the parabolic mirror onto patty paper first, then position it over a coordinate plane so they can easily locate the vertex at the origin and identify the coordinates of symmetric points.

As students work, look for

The equation used to represent the parabolic mirror in Question 3. Did students use $y^2 = 4px$ or did they use $(y - k)^2 = 4p(x - h)$? Can either equation be used to solve this problem situation? Is there an advantage to using one equation rather than the other?

Misconception

Students may incorrectly assume the location of the focus is a horizontal distance 6 inches to the right of the vertex. The given information of 6 inches should be used to identify the coordinates of the symmetric points (6, 36) and (6, −36) located at the upper right top edge and lower right bottom edge of the parabolic surface of the mirror.

Questions to ask

- Where is the vertex of the parabolic mirror in the diagram?
- How can you locate the parabolic mirror on a coordinate plane to provide coordinates that you can use to solve the problem?
- If you locate the vertex of the parabola at the origin, what are the coordinates of two symmetric points located at the top right and bottom right of the parabolic surface?
- If these two symmetric points are situated at a horizontal distance of 6 inches from the vertex, what are the x-coordinates of both points?
- How did you determine the y-coordinates of both points?
- Why are the coordinates of two symmetric points located at the top right and lower right of the parabolic surface (6, 36) and (6, −36)?
- What is the orientation of the parabola?
- Which standard equation, $y^2 = 4px$ or $x^2 = 4py$, is used to solve for the p-value?
- Why is the equation $y^2 = 4px$ appropriate in this situation?
- What point did you use to solve for the p-value?
- Is the p-value the solution to this problem situation?

Summary

The characteristics of parabolas can be used to solve real-world problems.

Talk the Talk: Simply Parabolic

DEMONSTRATE

Facilitation Notes

In this activity, students graph parabolas given equations. They then label the parabola with its key characteristics: the vertex, focus, and directrix. They then describe the concavity of each.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Questions to ask

- How is the equation in Question 2 different from the equation used in Question 1?
- What aspect of the equation indicates the location of the vertex?
- What aspect of the equation indicates a vertical or horizontal orientation?
- How was $y^2 = 4px$ used to determine the p-value?
- What aspect of the equation indicates the concavity?

Differentiation strategy

To extend the activity, ask students to summarize all of the connections between each of the characteristics of a parabola. Have them design a graphic organizer divided into spaces that contain different givens. To complete the organizer they must list the characteristics that are conclusive based on each scenario of different givens.

Summary

The value of p in the standard form of a parabola can be determined from the equation and used to determine the characteristics of a parabola—including the vertex, axis of symmetry, focus, directrix, and concavity.

NOTES

5

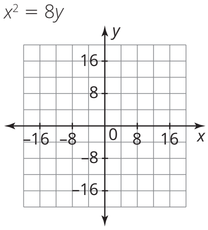
Going the Equidistance

Equation of a Parabola

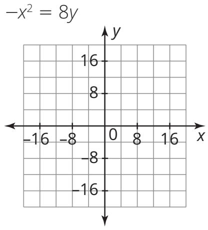
Warm Up

Use graphing technology to graph each equation and its inverse. Determine the vertex and axis of symmetry of each graph.

1. $x^2 = 8y$



2. $-x^2 = 8y$



Learning Goals

- Derive the equation of a parabola given the focus and the directrix.
- Solve problems using characteristics of parabolas.

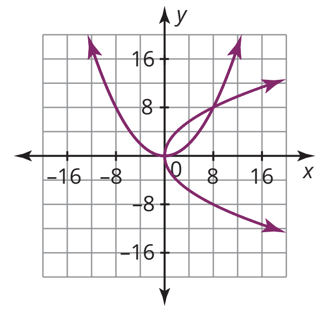
Key Terms

- locus of points
- focus
- directrix
- vertex of a parabola
- concavity
- general form of a parabola
- standard form of a parabola

You have studied parabolas as quadratic functions. What are the characteristics of a parabola determined by the set of points equidistant from both a given point and a given line?

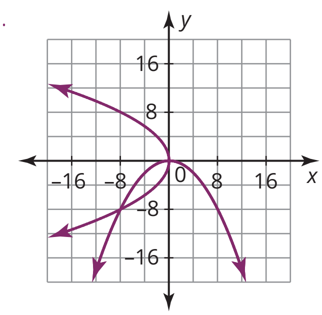
Warm Up Answers

1.



For $x^2 = 8y$, the vertex is (0, 0) and the axis of symmetry is $x = 0$.
For $y^2 = 8x$, the vertex is (0, 0) and the axis of symmetry is $y = 0$.

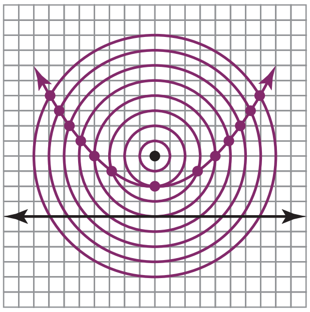
2.



For $-x^2 = 8y$, the vertex is (0, 0) and the axis of symmetry is $x = 0$.
For $-y^2 = 8x$, the vertex is (0, 0) and the axis of symmetry is $y = 0$.

Answers

1a.



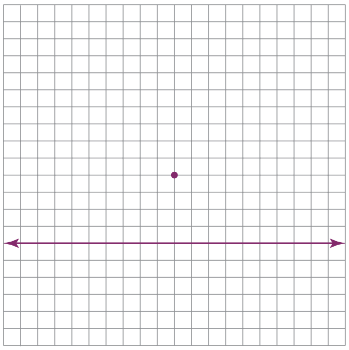
All the points are 1 unit away from the plotted point since it is the center of the circle.

1b. No. All the points that are 1 unit away from the graphed line lie along a line that is 1 unit above or below the graphed line. Neither line intersects with the circle, so there are no points on the circle that are the same distance from the plotted point and the graphed line.

GETTING STARTED

Hocus Pocus . . . Make the Locus!

Consider the grid with the plotted point and graphed line. Can you create a set of points that are the same distance from both the point and the line?



1. **Construct a circle with a radius of 1 unit using the point as the circle's center.**

   a. **What is the relationship between the points on the circle and the plotted point?**

   b. **Are any of the points on the circle the same distance from the plotted point and the graphed line? Explain your reasoning.**

2. **Construct a circle with a radius of 2 units using the point as the circle's center.**

a. Are any of the points on the circle the same distance from the plotted point and the graphed line? Explain your reasoning.

b. Plot a point on the circle that is the same distance from the circle's center and the line.

3. Continue to construct a total of eight concentric circles using the point as the center with the radius of each successive circle increasing by 1 unit.

a. How can you determine which points on each new circle are the same distance from the circle's center and the line?

b. Plot the points on each circle that are the same distance from the circle's center and the line.

Think about:

Will any of the points be below the graphed line?

4. Connect the points you plotted with a smooth curve.

a. What shape did you draw?

b. What do all the points on the curve have in common?

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Answers

2a. Yes; All the points on the circle are 2 units from the plotted point, and all points that are 2 units away from the graphed line lie along a line that is 2 units above or below the graphed line. The circle intersects with a line 2 units above the graphed line at one point.

2b. Check students' points.

3a. I can use the radius of each circle to determine the number of units away from the plotted point each circle is. I can then determine where the circle intersects the horizontal line that is the same number of units away from the graphed line.

3b. Check students' points.

4. See graph in Question 1.

4a. a parabola

4b. All the points on the curve are the same distance from the plotted point and the graphed line.

ELL Tip

Review the term *successive*. If students are unfamiliar with the term, define *successive* as *following one another, or following others*. Provide a list of synonyms for *successive* with which students may be familiar, such as *consecutive, sequential, in a row,* and *running*. Read aloud Question 3 and guide students to make the connection of the definition of *successive* and the use of the term in the phrase "...each *successive* circle..." in the context of the problem.

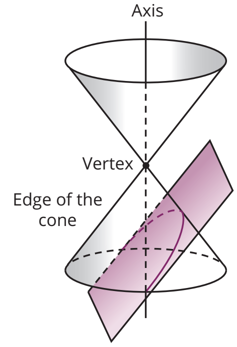
ACTIVITY
5.1

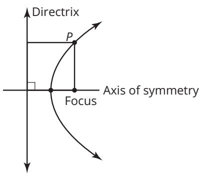
A Parabola as a Locus of Points

Everything you already know about parabolas remains true. The axis of symmetry of a parabola is a line that passes through the parabola and divides it into two symmetrical parts that are mirror images of each other.

You previously studied parabolas as quadratic functions and recognized the equation of a parabola as $y = x^2$. You analyzed equations and graphed parabolas based on the position of the vertex and additional points determined by using x-values on either side of the axis of symmetry.

Recall that when a plane intersects one nappe of the double-napped cone parallel to the edge of the cone, the curve that results is a parabola.





In this lesson, you will explore a parabola as a *locus of points* to determine more information about the parabola. A **locus of points** is a set of points that share a property. A parabola is the set of all points in a plane that are equidistant from both a *focus* and a *directrix*. The **focus** is a point which lies inside the parabola on the axis of symmetry. The **directrix** is a line that is perpendicular to the axis of symmetry and lies outside the parabola and does not intersect the parabola.

4 • TOPIC 3: Applications of Quadratics

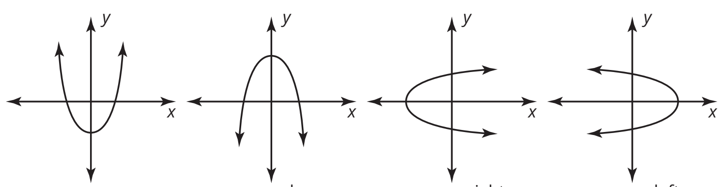
1. Create a parabola by folding patty paper.

a. Take a piece of patty paper. Near the bottom of the paper, draw a line (the directrix). Draw a point (the focus) above the line and label it point *F*.

b. Fold the patty paper so that point *F* and the line meet and make a crease. Next, slide point *F* along the line continuing to crease the patty paper. Repeat this process for at least 30 points along the line.

c. What conic section is formed by the folds? Outline the shape.

The **concavity** of a parabola describes the orientation of the curvature of the parabola. A parabola can be concave up, concave down, concave right, or concave left, as shown. The **vertex of a parabola** is the point on the axis of symmetry which is exactly midway between the focus and the directrix. It is also the point where the parabola changes direction.

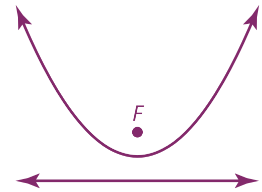


concave up | concave down | concave right | concave left

2. Label the vertex of the parabola you constructed.

Answers

1a-c. Check students' patty paper.
Sample answer.

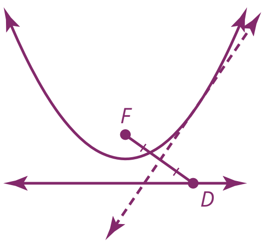


The folds form a parabola.

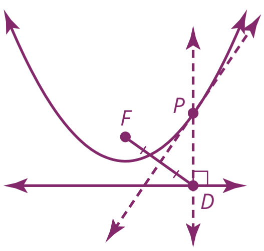
2. Check students' patty paper.

Answers

3a. The line formed by the crease is the perpendicular bisector of $\overline{FD}$.



3b. Point *P* is on the parabola.



3c. The distance from point *P* to point *F* is equal to the distance from point *P* to point *D* because point *P* is on the perpendicular bisector. The definition of a perpendicular bisector is that any point on the perpendicular bisector is equal distance from the two endpoints.

3d. Sample answer. No matter what point I pick on the directrix, point *P* is on the parabola, and it is an equal distance from the focus and the directrix.

4. Any point on a parabola is an equal distance from the focus and the directrix.

3. Investigate the parabola you constructed using patty paper.

a. Draw a point on the directrix and label it point *D*. Draw a line from point *F*, the focus, to point *D*. Fold the paper so points *F* and *D* meet. What is the relationship between the line formed by the crease and $\overline{FD}$?

b. Draw a line perpendicular to the directrix through point *D*. Label the point where the perpendicular line intersects the crease as point *P*. Where does point *P* lie?

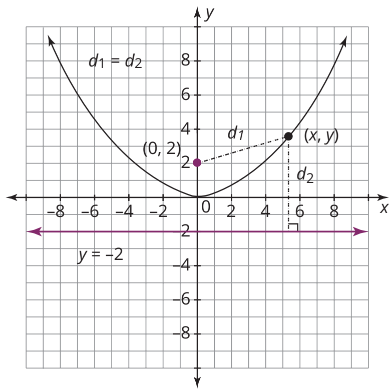
c. What is true about the distances from point *P* to point *F* and to point *D*? How do you know this is true?

d. Draw two additional points on the directrix and repeat parts (a) through (c). What do you notice?

4. Summarize what you discovered about the distance from any point on a parabola to the focus and directrix.

ACTIVITY 5.2 Equations of a Parabola

The parabola shown is defined such that all points on the parabola are equidistant from the point (0, 2) and the line $y = -2$. One point on the parabola is labeled as (x, y).



1. Determine the equation of the parabola by completing the steps.

a. Let d_1 represent the distance from (x, y) to (0, 2). Write an equation using the Distance Formula to represent d_1. Simplify the equation.

b. Let d_2 represent the distance from (x, y) to the line $y = -2$. Write an equation using the Distance Formula to represent d_2. Simplify the equation.

c. What do you know about the relationship between d_1 and d_2?

d. Write an equation for the parabola using Question 1, parts (a) through (c). Simplify the equation so that one side of the equation is x^2.

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Answers

1a. $d_1 = \sqrt{(x-0)^2 + (y-2)^2}$
$= \sqrt{x^2 + (y-2)^2}$

1b. $d_2 = \sqrt{(x-x)^2 + (y+2)^2}$
$= \sqrt{(y+2)^2}$

1c. The two distances are equal to one another.

1d. $\sqrt{x^2 + (y-2)^2} = \sqrt{(y+2)^2}$

$(\sqrt{x^2 + (y-2)^2})^2 = (\sqrt{(y+2)^2})^2$

$x^2 + (y-2)^2 = (y+2)^2$

$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$

$x^2 - 4y = 4y$

$x^2 = 8y$

Answers

2. The standard form of the equation is $x^2 = 8y$. The general form of the equation is $x^2 - 8y = 0$.
3. The x-intercept of the parabola is the point (0, 0). The y-intercept of the parabola is the point (0, 0).
4. The parabola is symmetric about the y-axis.
5. The point is (4, 2).
6. (−4, 2); the point will have the same y-coordinate and be the same distance from the y-axis in the opposite direction.
7. $x^2 = 8y$

   $x^2 = 8(4.5)$

   $x^2 = 36$

   $\sqrt{x^2} = \sqrt{36}$

   $x = 6$ and $x = -6$

   The points are (6, 4.5) and (−6, 4.5).

The **general form of a parabola** with a vertex at the origin is an equation of the form $Ax^2 + Dy = 0$ or $By^2 + Cx = 0$.

The **standard form of a parabola** with a vertex at the origin is an equation of the form $x^2 = 4py$ or $y^2 = 4px$, where p represents the distance from the vertex to the focus.

2. Write the equation of the parabola from Question 1 in general form and in standard form.

3. What are the coordinates for the intercept(s) of the parabola?

4. Describe the symmetry of the parabola.

5. Calculate the coordinates of the point that has an *x*-coordinate of 4.

6. Use symmetry to determine which other point on the parabola has a *y*-coordinate of 2. Explain your reasoning.

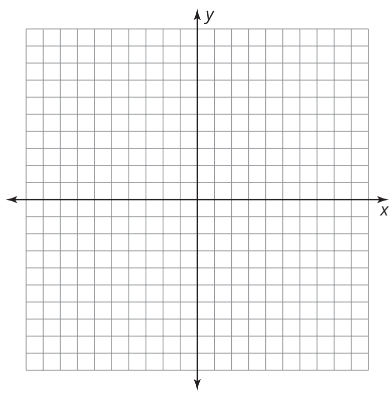
7. Calculate the coordinates of each point that has a *y*-coordinate of 4.5.

8. How can you determine the points on the parabola that have a *y*-coordinate of −4.5 graphically and algebraically?

9. Consider the parabola represented by the equation $y^2 = 2x$.

a. Sketch the parabola using the table of values.

x	y
8	−4
2	−2
0	0
2	2
8	4



Ask yourself:

What does the form of the equation tell you about the shape of the parabola?

b. What is the relationship between the axis of symmetry and the equation of the parabola?

c. What are the coordinates of the vertex?

d. How is the concavity of the parabola related to the orientation of the parabola?

Answers

8. $x^2 = 8y$

$x^2 = 8(-4.5)$

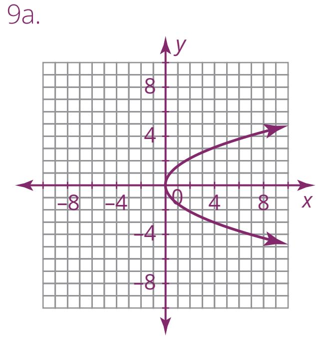
$x^2 = -36$

$\sqrt{x^2} = \sqrt{-36}$

The square root of a negative number is an imaginary number so there are no points on the parabola that have a *y*-coordinate of −4.5.

I can also look at the graph to see that the parabola is concave up and will have no *y*-values less than the minimum *y*-value of the vertex.

9a.



9b. The parabola has a horizontal orientation. Its axis of symmetry is the *x*-axis represented by the line $y = 0$. The parabola's equation has a y^2 term.

9c. The vertex of the parabola is (0, 0).

9d. A parabola that has a horizontal orientation is either concave left or concave right and the graphed parabola is concave right.

Answers

10a. See the table and graph below.

10b. The parabola has a vertical orientation. Its axis of symmetry is the *y*-axis represented by the line $x = 0$. The parabola's equation has an x^2 term.

10c. The vertex of the parabola is (0, 0).

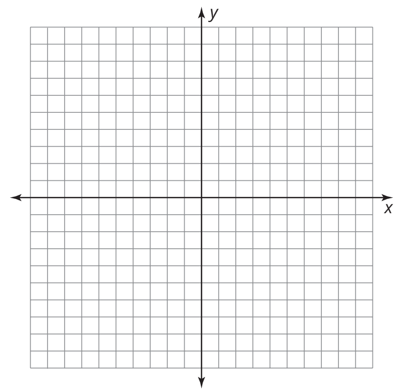
10d. A parabola that has a vertical orientation is either concave up or concave down, and the graphed parabola is concave up.

11a. If the axis of symmetry is along the *y*-axis, then the standard form of the parabola is $x^2 = 4py$.

10. Consider the parabola represented by the equation $x^2 = 9y$.

a. Complete the table of values for the equation. Sketch the parabola using the coordinates from the table.

x	y
−6	
−3	
0	
3	
6	



b. What is the relationship between the axis of symmetry and the equation of the parabola?

c. What are the coordinates of the vertex?

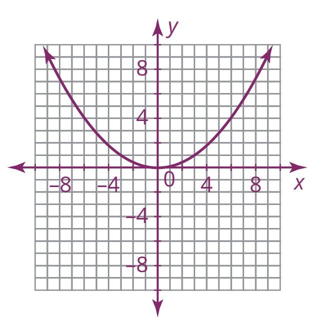
d. How is the concavity of the parabola related to the orientation of the parabola?

11. The standard form of a parabola with its vertex at the origin is an equation of the form $x^2 = 4py$ or $y^2 = 4px$.

a. What is the standard form of a parabola with an axis of symmetry along the *y*-axis?

10a.

x	y
−6	4
−3	1
0	0
3	1
6	4



b. **What is the standard form of a parabola with an axis of symmetry along the *x*-axis?**

c. **What is the equation of the axis of symmetry for a parabola with a vertical orientation?**

d. **What is the equation of the axis of symmetry for a parabola with a horizontal orientation?**

e. **Is the concavity of a parabola with a vertical orientation described as concave up/down or concave right/left?**

f. **Is the concavity of a parabola with a horizontal orientation described as concave up/down or concave right/left?**

Answers

11b. If the axis of symmetry is along the *x*-axis, then the standard form of the parabola is $y^2 = 4px$.

11c. The equation of the axis of symmetry for a parabola with a vertical orientation is $x = 0$.

11d. The equation of the axis of symmetry for a parabola with a horizontal orientation is $y = 0$.

11e. The concavity of a parabola with a vertical orientation would be described as either concave up or concave down.

11f. The concavity of a parabola with a horizontal orientation would be described as either concave left or concave right.

Answers

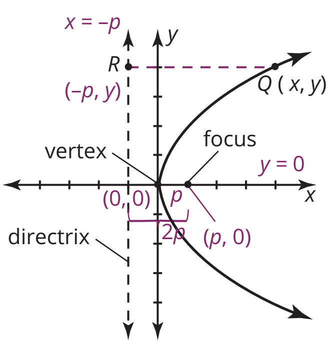
1a. The coordinates for the vertex are (0, 0). See graph below.

1b. See graph below.

1c. The coordinates of the focus are $(p, 0)$. See graph below.

1d. The equation for the directrix is $x = -p$. By definition, any point on the parabola is equidistant between the focus and the directrix. The vertex is p units from the focus, so it must be p units from the directrix. Because the directrix is on the other side of the vertex, opposite the focus, the point on the directrix that is on the line of symmetry is $(-p, 0)$. The directrix is perpendicular to the axis of symmetry, so it is a vertical line, and its equation is $x = -p$.

1e. The distance from the focus to the directrix is $2p$. See graph below.

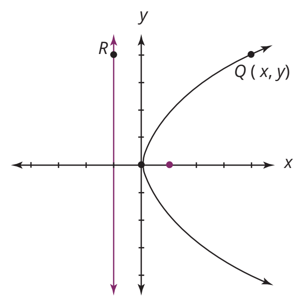


ACTIVITY
5.3

Making Sense of the Constant *p*

You have learned that the standard form of a parabola with a vertex at the origin is an equation of the form $x^2 = 4py$ or $y^2 = 4px$, where p represents the distance from the vertex to the focus. What is the significance of the value $4p$ in each equation?

1. Consider the sketch of the parabola. Let *p* represent the distance from the vertex to the focus.



a. Label the vertex with its coordinates.

b. Label the distance, *p*, on the graph.

c. Label the focus with its coordinates.

d. Label the directrix with the equation for its line. Explain your reasoning.

e. What is the distance from the focus to the directrix? Label this distance on the graph.

Think about:

What is the relationship between the directrix and the axis of symmetry?

2. **Write an equation for the parabola.**

   a. **Let d_1 represent the distance from point *Q* on the parabola to the focus. Write an equation using the Distance Formula to represent d_1. Simplify the equation.**

   b. **Line segment *QR* represents the perpendicular distance from point *Q* on the parabola to the directrix. Draw line segment *QR*. What are the coordinates of point *R*? Label the coordinates on the graph.**

   c. **Let d_2 represent the distance from point *Q* to point *R*. Write an equation using the Distance Formula to represent d_2. Simplify the equation.**

   d. **Write an equation for the parabola using parts (a) through (c). Simplify the equation so that one side of the equation is the squared term.**

   e. **Describe the significance of the equation derived in part (d).**

Answers

2a. The points are (x, y) and $(p, 0)$.

$d_1 = \sqrt{(x - p)^2 + (y - 0)^2}$

$d_1 = \sqrt{(x - p)^2 + y^2}$

2b. The coordinates of point *R* are $(-p, y)$. See graph on the previous page.

2c. The points are (x, y) and $(-p, y)$.

$d_2 = \sqrt{(x + p)^2 + (y - y)^2}$

$d_2 = \sqrt{(x + p)^2}$

2d. $\sqrt{(x - p)^2 + y^2} = \sqrt{(x + p)^2}$

$(\sqrt{(x - p)^2 + y^2})^2 = (\sqrt{(x + p)^2})^2$

$(x - p)^2 + y^2 = (x + p)^2$

$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$

$-2px + y^2 = 2px$

$y^2 = 4px$

2e. Sample answer. The equation is the standard form of a parabola with the *x*-axis as the axis of symmetry. The development of the equation explains why the equation includes $4p$. The value of p can be determined from the equation and then used to place the focus and directrix on the graph.

Answers

1a. The vertex of the parabola is (0, 0). The axis of symmetry is the x-axis represented by the line $y = 0$.

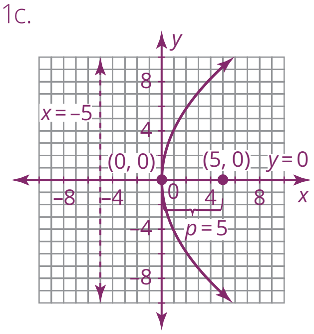
1b. $y^2 = 4px$

$y^2 = 20x$

So, $4p = 20$ and $p = 5$.

The coordinates of the focus are (5, 0). The focus is 5 units from the vertex along the x-axis.
The equation of the directrix is $x = -5$. The directrix is perpendicular to the axis of symmetry and 5 units from the vertex in the opposite direction of the focus.

1c.



The parabola is concave right. The parabola opens toward the focus and away from the directrix.

ACTIVITY **5.4**

Using the Constant p to Graph a Parabola

Let's investigate different equations to understand the relationships between the structure of the equation and its graph.

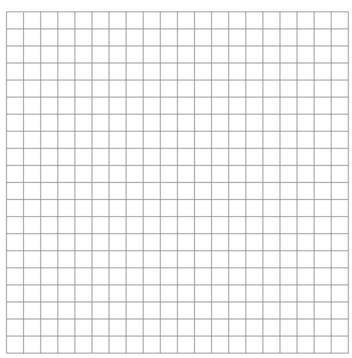
Ask yourself:

What does the form of the equation tell you? What do you think the graph will look like?

1. Consider the parabola represented by the equation $y^2 = 20x$.

a. Identify the coordinates of the vertex and the equation of the line of symmetry.

b. Determine the value of p. Then determine the coordinates of the focus and the equation of the directrix. Justify your reasoning.



c. Graph the parabola. Then describe the concavity of the parabola. Justify your reasoning.

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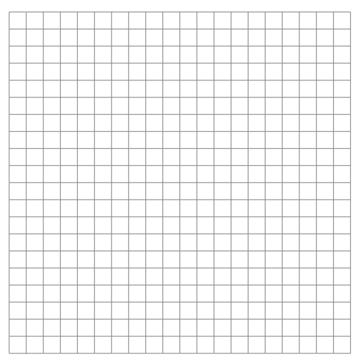
2. **Consider the parabola represented by the equation $x^2 = -12y$.**

   a. **Identify the coordinates of the vertex and the equation of the line of symmetry.**

   b. **Determine the value of p. Then determine the coordinates of the focus and the equation of the directrix. Justify your reasoning.**

   c. **Graph the parabola. Then describe the concavity of the parabola. Justify your reasoning.**



Answers

2a. The vertex of the parabola is (0, 0).
The axis of symmetry is the y-axis represented by the line $x = 0$.

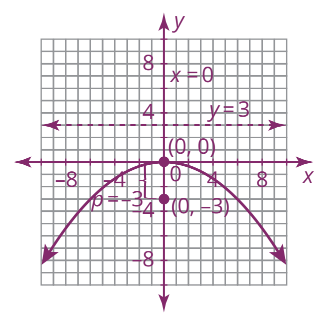
2b. $x^2 = 4py$

$x^2 = -12y$

So, $4p = -12$ and $p = -3$.

The coordinates of the focus are (0, −3). The focus is −3 units from the vertex along the y-axis.
The equation for the directrix is $y = 3$.
The directrix is perpendicular to the axis of symmetry and 3 units from the vertex in the opposite direction of the focus.

2c.



The parabola is concave down. The parabola opens toward the focus and away from the directrix.

Answers

3a. The vertex of the parabola is (0, 0). The axis of symmetry is the y-axis represented by the line $x = 0$.

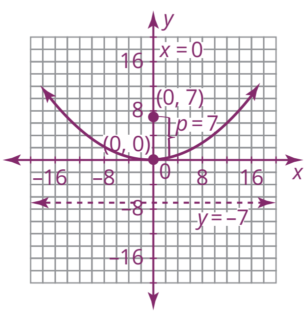
3b. $x^2 = 4py$

$x^2 = 28y$

So, $4p = 28$ and $p = 7$.

The coordinates of the focus are (0, 7). The focus is 7 units from the vertex along the y-axis.
The equation for the directrix is $y = -7$. The directrix is perpendicular to the axis of symmetry and -7 units from the vertex in the opposite direction of the focus.

3c.

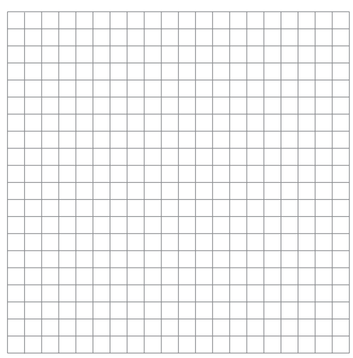


The parabola is concave up. The parabola opens toward the focus and away from the directrix.

3. Consider the parabola represented by the equation $x^2 = 28y$.

a. Identify the coordinates of the vertex and the equation of the line of symmetry.

b. Determine the value of p. Then determine the coordinates of the focus and the equation of the directrix. Justify your reasoning.



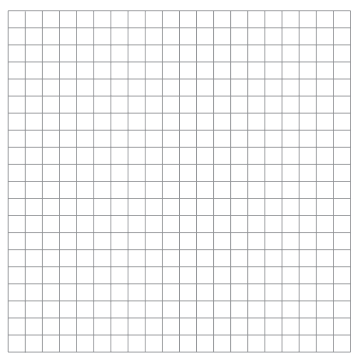
c. Graph the parabola. Then describe the concavity of the parabola. Justify your reasoning.

4. **Consider the parabola represented by the equation $y^2 = -10x$.**

   a. **Identify the coordinates of the vertex and the equation of the line of symmetry.**

   b. **Determine the value of *p*. Then determine the coordinates of the focus and the equation of the directrix. Justify your reasoning.**

   c. **Graph the parabola. Then describe the concavity of the parabola. Justify your reasoning.**



Answers

4a. The vertex of the parabola is (0, 0).
The axis of symmetry is the *x*-axis represented by the line $y = 0$.

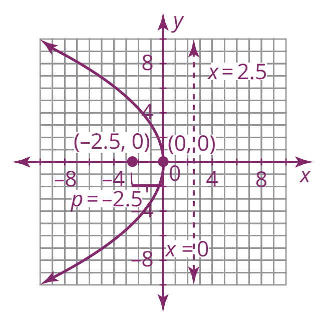
4b. $y^2 = 4px$

$y^2 = -10x$

So, $4p = -10$ and $p = -2.5$.

The coordinates of the focus are (−2.5, 0). The focus is −2.5 units from the vertex along the *x*-axis.
The equation of the directrix is $x = 2.5$. The directrix is perpendicular to the axis of symmetry and 2.5 units from the vertex in the opposite direction of the focus.

4c.



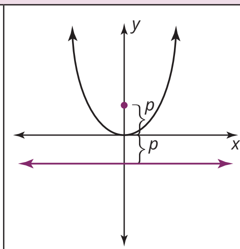
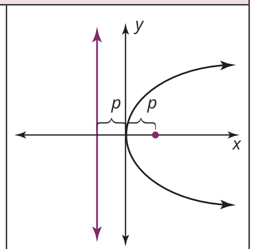
The parabola is concave left. The parabola opens toward the focus and away from the directrix.

Answers

5. When p is positive, the parabola is concave up or concave right. When p is negative, the parabola is concave down or concave left.
6. See partial table below.

5. Analyze each equation and its corresponding graph in Questions 1 through 4. Describe the relationship between the sign of the constant *p* and the concavity of each parabola.

6. Complete the table.

Parabola Centered at Origin		
Graph		
Equation of Parabola		
Orientation of Parabola		
Axis of Symmetry		
Coordinates of Vertex		
Coordinates of Focus		
Equation of Directrix		
Concavity		

6.

Equation of Parabola	$x^2 = 4py$	$y^2 = 4px$
Orientation of Parabola	vertical	horizontal
Axis of Symmetry	y-axis, $x = 0$	x-axis, $y = 0$
Coordinates of Vertex	(0, 0)	(0, 0)
Coordinates of Focus	$(0, p)$	$(p, 0)$
Equation of Directrix	$y = -p$	$x = -p$
Concavity	$p > 0$ concave up $p < 0$ concave down	$p > 0$ concave right $p < 0$ concave left

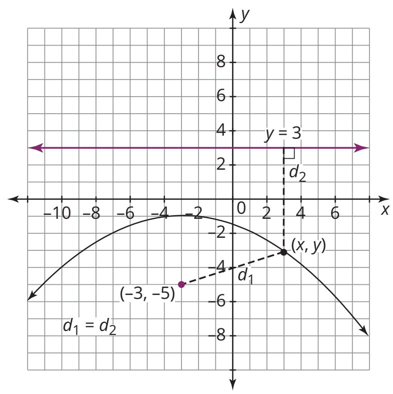
ACTIVITY **5.5**

Writing an Equation Given a Focus and a Directrix

In this activity, you will use the Distance Formula to determine the equation of points that are equidistant from a given point (the focus) and a given line (the directrix) where the vertex is a point other than the origin.

1. Consider the graph shown.

a. Determine an equation for all the points equidistant from the point (−3, −5) and the line $y = 3$.



b. Determine the coordinates of the vertex and equation of the axis of symmetry of the parabola. Explain your reasoning.

Answers

1a. $\sqrt{(x+3)^2 + (y+5)^2} = \sqrt{(x-x)^2 + (y-3)^2}$

$\left(\sqrt{(x+3)^2 + (y+5)^2}\right)^2 = \left(\sqrt{(y-3)^2}\right)^2$

$(x+3)^2 + (y+5)^2 = (y-3)^2$

$x^2 + 6x + 9 + y^2 + 10y + 25 = y^2 - 6y + 9$

$x^2 + 6x + 9 = -16y - 16$

$x^2 + 6x + 16y + 25 = 0$

1b. Since the focus is the point (−3, −5) and the directrix is the line $y = 3$, I know the axis of symmetry is the vertical line through the x-coordinate of the focus, or $x = -3$. The vertex of the parabola is an equal distance between the focus and the directrix along the axis of symmetry. Since the total distance between the focus and the directrix along the axis of symmetry is 8 units, the vertex is 4 units away from both the focus and the directrix along the same line. So the vertex is the point (−3, −1).

Answers

2a. $\sqrt{(x-7)^2+(y-5)^2} = \sqrt{(x-1)^2+(y-y)^2}$

$(\sqrt{(x-7)^2+(y-5)^2})^2 = (\sqrt{(x-1)^2})^2$

$(x-7)^2+(y-5)^2 = (x-1)^2$

$x^2-14x+49+y^2-10y+25 = x^2-2x+1$

$y^2-12x-10y+73=0$

2b. Since the focus is the point (7, 5) and the directrix is the line $x = 1$, I know the axis of symmetry is the horizontal line through the y-coordinate of the focus, or $y = 5$. The vertex of the parabola is an equal distance between the focus and the directrix along the axis of symmetry. Since the total distance between the focus and the directrix along the axis of symmetry is 6 units, the vertex is 3 units away from both the focus and the directrix along the same line. So the vertex is the point (4, 5).

3. $x^2+6x+16y+25=0$

$x^2+6x=-16y-25$

$x^2+6x+9=-16y-25+9$

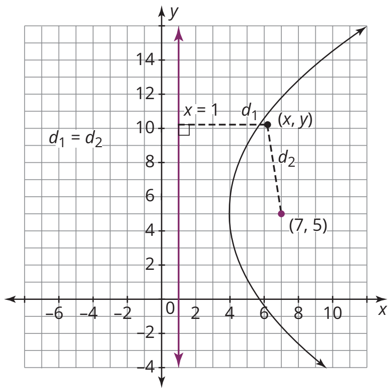
$(x+3)^2=-16(y+1)$

$y^2-12x-10y+73=0$

$y^2-10y=12x-73$

$y^2-10y+25=12x-73+25$

$(y-5)^2=12(x-4)$



2. Consider the graph shown.

a. Determine the equation for all the points equidistant from the point (7, 5) and the line $x = 1$.

b. Determine the coordinates of the vertex and equation of the axis of symmetry of the parabola. Explain your reasoning.

The standard forms of parabolas with vertex at (h, k) are $(x-h)^2 = 4p(y-k)$ and $(y-k)^2 = 4p(x-h)$.

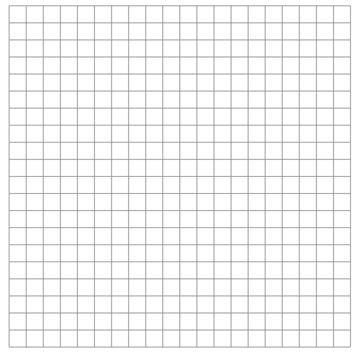
Think about:

How do these equations compare with the standard form of the equation of a circle: $(x-h)^2+(y-k)^2=r^2$?

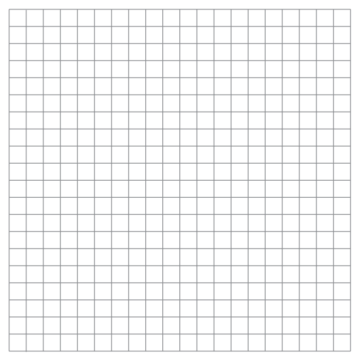
3. Rewrite the equations from Questions 1 and 2 in one of these forms.

4. Rewrite each equation in standard form. Determine the value of *p*, the coordinates of the vertex and focus, and the equations of the directrix and the axis of symmetry. Then sketch the graph of the parabola with the focus and directrix and describe the concavity.

a. $y^2 + 8y + 8x + 16 = 0$



b. $4x^2 - 40x + 48y + 4 = 0$



Answers

4a. $y^2 + 8y + 8x + 16 = 0$

$y^2 + 8y = -8x - 16$

$y^2 + 8y + 16 = -8x - 16 + 16$

$(y + 4)^2 = -8x$

$p = -2$

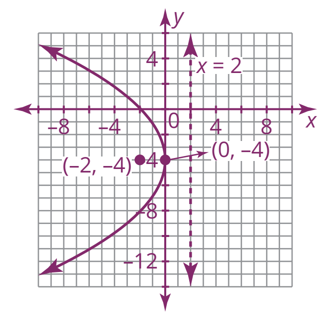
Vertex: (0, −4)

Focus: (−2, −4)

Directrix: $x = 2$

Axis of symmetry: $y = -4$

Concavity: concave left



4b. $4x^2 - 40x + 48y + 4 = 0$

$x^2 - 10x + 12y + 1 = 0$

$x^2 - 10x = -12y - 1$

$x^2 - 10x + 25 = -12y - 1 + 25$

$(x - 5)^2 = -12(y - 2)$

$p = -3$

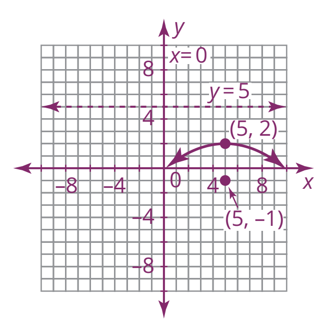
Vertex: (5, 2)

Focus: (5, −1)

Directrix: $y = 5$

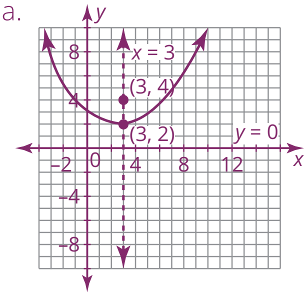
Axis of symmetry: $x = 5$

Concavity: concave down



Answers

5a.



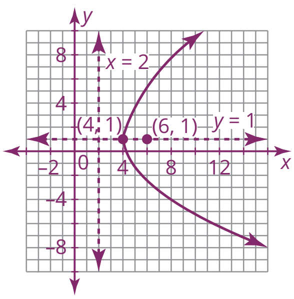
The focus is 2 units above the vertex, so $p = 2$.

The standard form of a parabola with a vertical axis of symmetry is $(x - h)^2 = 4p(y - k)$. By substitution,

$(x - 3)^2 = 4(2)(y - 2)$.

So, the equation is $(x - 3)^2 = 8(y - 2)$.

5b.



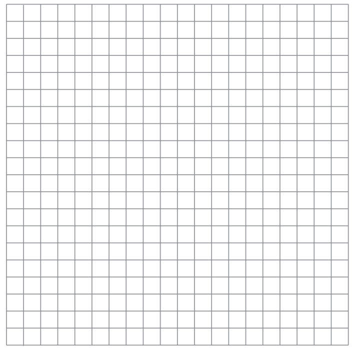
The focus is 2 units to the right of the vertex, so $p = 2$.

The standard form of a parabola with a horizontal axis of symmetry is $(y - k)^2 = 4p(x - h)$. By substitution,

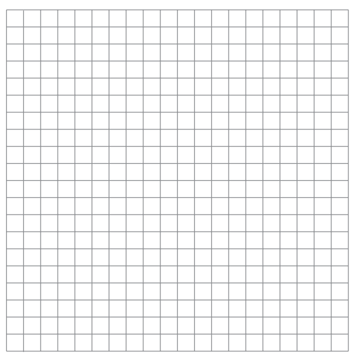
$(y - 1)^2 = 4(2)(x - 4)$.

So, the equation is $(y - 1)^2 = 8(x - 4)$.

5. Write an equation in standard form for each parabola. Then, graph and label the parabola.



a. A parabola with a vertex at (3, 2) and a focus at (3, 4).



b. A parabola with a vertex at (4, 1) and a directrix at $x = 2$.

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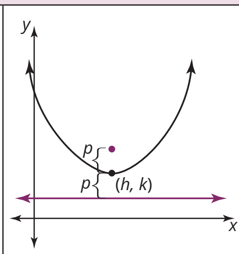
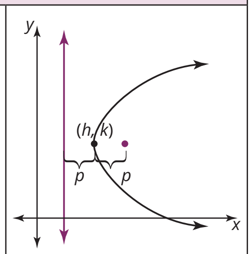
Answer

6. See table below.

6. Complete the table.

Parabola		
Graph		
Equation of Parabola	$(x - h)^2 = 4p(y - k)$	$(y - k)^2 = 4p(x - h)$
Orientation of Parabola		
Axis of Symmetry		
Coordinates of Vertex		
Coordinates of Focus		
Equation of Directrix		
Concavity		

6.

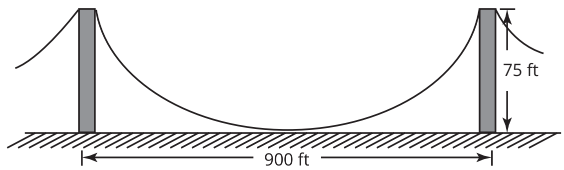
Orientation of Parabola	vertical	horizontal
Axis of Symmetry	$x = h$	$y = k$
Coordinates of Vertex	(h, k)	(h, k)
Coordinates of Focus	$(h, k + p)$	$(h + p, k)$
Equation of Directrix	$y = k - p$	$x = h - p$
Concavity	$p > 0$ concave up $p < 0$ concave down	$p > 0$ concave right $p < 0$ concave left

Answers

1. Place the bridge on a coordinate plane with the center of the bridge at the origin. Three coordinate pairs are known: (0, 0), (450, 75) and (−450, 75).
   The coordinates of the vertex (0, 0) and another point (450, 75) can be substituted into the standard form of a parabola with a vertical axis of symmetry to calculate the value of p.
   The standard form of a parabola with a vertical axis of symmetry is $(x - h)^2 = 4p(y - k)$.
   Substitute the coordinates of the vertex:
   $(x - 0)^2 = 4(p)(y - 0)$
   $x^2 = 4py$
   Substitute the point (450, 75):
   $450^2 = 4p(75)$
   $450^2 = 300p$
   $202{,}500 = 300p$
   $675 = p$
   $x^2 = 4(675)y$
   $x^2 = 2700y$
2. Place the parabola that represents the satellite dish on a coordinate plane with its vertex at the origin. Three coordinate pairs are known: (0, 0), (−2.5, 1) and (2.5, 1).
   The equation in standard form for a parabola with its center at the origin is $x^2 = 4py$.
   Substitute the point (2.5, 1): $x^2 = 4py$
   $(2.5)^2 = 4p(1)$
   $6.25 = 4p$
   $1.5625 = p$
   The distance from the vertex to the focus should be 1.5625 feet. So, the receiver should be placed 1.5625 feet away from the vertex of the satellite dish.

ACTIVITY **5.6** Applications of Parabolas

1. The main cables of a suspension bridge are parabolic. The parabolic shape allows the cables to bear the weight of the bridge evenly. The distance between the towers is 900 feet and the height of each tower is about 75 feet. Write an equation for the parabola that represents the cable between the two towers.

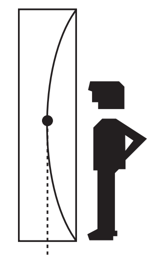


2. A cross-section of a satellite dish is a parabola. The satellite dish is 5 feet wide at its opening and 1 foot deep. The receiver of the satellite dish should be placed at the focus of the parabola. How far should the receiver be placed from the vertex of the satellite dish?

ELL Tip

Assess students' prior knowledge of the term *suspension bridge*. If they are unfamiliar, first discuss the term *suspension*. Define the root word *suspend*, in the context of the term *suspension bridge,* as *to hang something from somewhere*. Sketch a drawing of the bridge in Question 1. Define a *suspension bridge* as *a bridge that is supported by larger cables that are suspended and anchored between towers*.

3. Many carnivals and amusement parks have mirrors that are parabolic. When you look at your reflection in a parabolic mirror, your image appears distorted and makes you look taller or shorter depending on the shape of the mirror. The focal length of a mirror is the distance from the vertex to the focus of the mirror. Consider a mirror that is 72 inches tall with a vertex that is 6 inches from the top and bottom edges of the mirror. What is the focal length of the mirror?



Answer

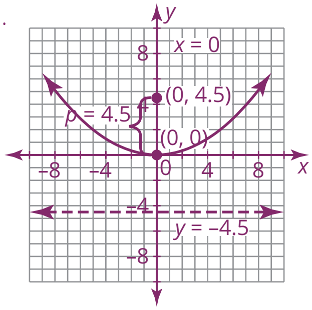
3. Place the parabola that represents the mirror on the coordinate plane with its vertex at the origin.
   Three coordinate pairs are known: (0, 0), (6, 36) and (6, −36).
   The coordinates of the vertex (0, 0) and another point (6, 36) can be substituted into the standard form of a parabola with a horizontal axis of symmetry to calculate the value of p. The standard form of a parabola with a horizontal axis of symmetry is
   $(y - k)^2 = 4p(x - h)$
   Substitute the vertex:
   $(y - 0)^2 = 4p(x - 0)$.
   $y^2 = 4px$
   Substitute the point (6, 36):
   $36^2 = 4p(6)$
   $36^2 = 24p$
   $1296 = 24p$
   $54 = p$
   The focal length of the mirror is 54 inches.

ELL Tip

Two non-mathematical terms that appear in this problem are *carnival* and *distorted*. Define a *carnival* as *a typically outdoor entertainment event with games and shows*. Define *distort* as *to cause something to change from its regular shape*. Read aloud the first two sentences of Question 3, "Many *carnivals* and amusement parks have mirrors that are parabolic. When you look at your reflection in a parabolic mirror, your image appears *distorted* and makes you look taller or shorter depending on the shape of the mirror." Clarify any further misunderstandings students may have about the use of the terms *carnival* and *distorted* in the context of the problem.

Answers

1.



Vertex: (0, 0)

Axis of symmetry: *y*-axis, $x = 0$

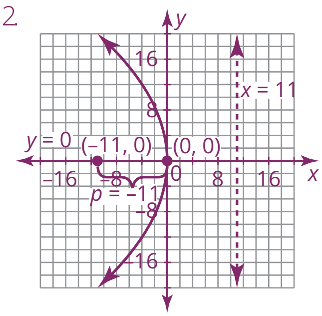
p: 4.5

Focus: (0, 4.5)

Directrix: $y = -4.5$

Concavity: concave up

2.



Standard form: $y^2 = -44x$

Vertex: (0, 0)

Axis of symmetry: *x*-axis, $y = 0$

p: −11

Focus: (−11, 0)

Directrix: $x = 11$

Concavity: concave left

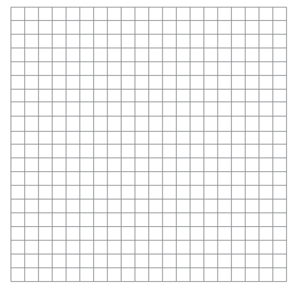
NOTES

TALK the TALK

Simply Parabolic

Graph each parabola. Label the vertex, the focus, and the directrix. Then describe the concavity.

1. $x^2 = 18y$



2. $y^2 + 44x = 0$