Length, Width, and Depth

4

Deepening Understanding of Volume

WARM UP

Determine each least common multiple.

1. LCM(2, 10)
2. LCM(3, 8)
3. LCM(6, 14)
4. LCM(10, 15)

LEARNING GOALS

- Determine the volume of right rectangular prisms with fractional edge lengths using unit cubes with unit fractional dimensions.
- Determine the number of cubes with unit fractional dimensions that have the same volume as a unit cube.
- Determine the number of unit cubes with unit fractional dimensions that can pack a rectangular prism with fractional edge lengths.
- Connect the volume formulas $V = lwh$ and $V = Bh$ with a unit-cube model of volume for rectangular prisms.

KEY TERMS

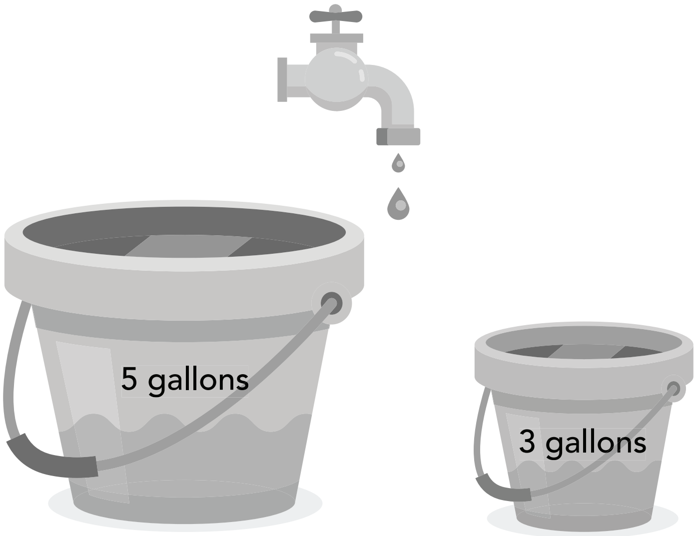
- volume
- geometric solid
- polyhedron
- face
- edge
- vertex
- unit cube
- right rectangular prism
- cube

You know about three-dimensional figures such as cubes and other rectangular prisms. You also know how to operate with positive rational numbers. How can you calculate the measurements of any rectangular prism, even one with fractional edge lengths?

Getting Started

Measuring Water

You have two empty containers, each with a different volume, as shown. You also have a source of water.



(1155 in.^{3}) (693 in.^{3})

Remember...

Volume is the amount of space occupied by an object. You measure the volume of an object in cubic units.

1. Using just these containers, how can you measure out a volume of exactly 4 gallons (924 in.^{3})?

ACTIVITY
4.1

Volume of Prisms

Recall that a polygon is a closed figure formed by three or more line segments. A **geometric solid** is a bounded three-dimensional geometric figure. A **polyhedron** is a three-dimensional solid figure made up of polygons. A **face** is one of the polygons that makes up a polyhedron. An **edge** is the intersection of two faces of a three-dimensional figure. The point where multiple edges meet is known as a **vertex** of a three-dimensional figure.

Figure A

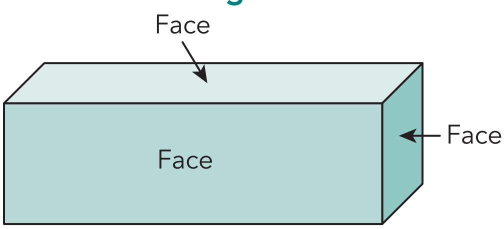
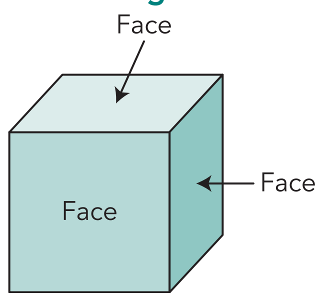


Figure B

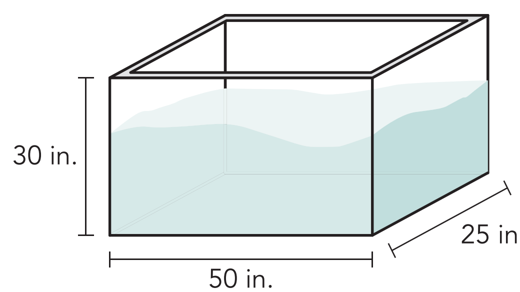


Did You Know?

A **unit cube** is a cube whose sides are all 1 unit long.

Figure A is a *right rectangular prism.* A **right rectangular prism** is a polyhedron with three pairs of congruent and parallel rectangular faces. Figure B is an example of a *cube*, which is a special kind of right rectangular prism. A **cube** is a polyhedron that has congruent squares as faces.

1. Consider the fish tank shown.



a. How can you determine the volume of the tank?

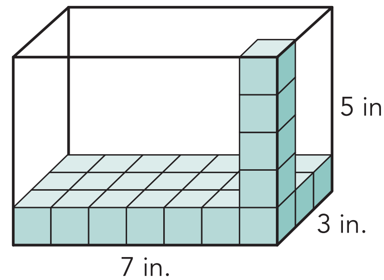
b. How many cubic inches of water can the tank hold?

2. A planter box in the shape of a rectangular prism has a length of 3 feet, a width of 1 foot, and a height of $1\frac{1}{2}$ feet. How much potting soil can the planter box hold?

Recall that you can calculate the volume of a rectangular prism by packing it with unit cubes.

WORKED EXAMPLE

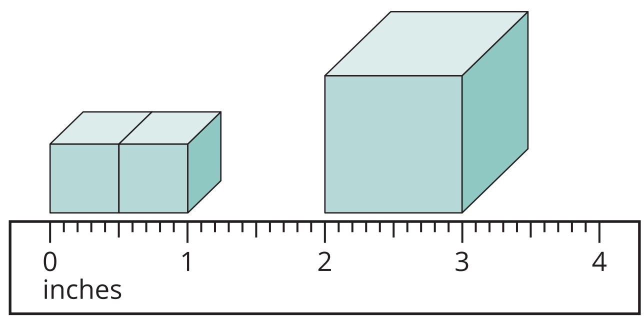
The prism shown can be packed with 7 unit cubes along its length, 3 unit cubes along its width, and 5 unit cubes along its height.



There are 7×3, or 21, unit cubes in each layer and there are 5 layers of cubes.

So, the prism can be packed with a total of 21×5, or 105, unit cubes. It has a volume of 105 cubic inches.

Consider a cube that has side lengths that are each $\frac{1}{2}$ inch.



Each side of the $\frac{1}{2}$-inch cube is half the length of a unit cube, so it takes two of these cubes to measure the same length as a 1-inch cube.

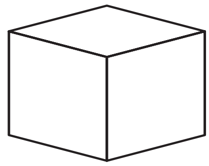
Consider filling the prism from the Worked Example with $\frac{1}{2}$-inch cubes.

3. Phillip says it will take twice as many cubes to fill the prism. Mara says it will take four times as many cubes to fill the prism. Daniel says it will take eight times as many cubes to fill the prism. Who is correct?

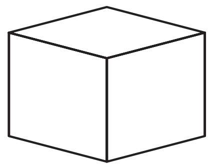
NOTES

4. **Determine how many of each fractional unit cube it takes to fill a unit cube. Draw the fractional unit cubes on the empty unit cube. Then determine the volume of each fractional unit cube.**

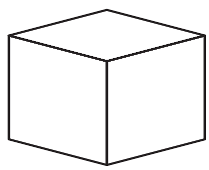
   a. **$\frac{1}{2}$-unit cube**



   b. **$\frac{1}{4}$-unit cube**



   c. **$\frac{1}{3}$-unit cube**



5. **How did you determine the number of fractional unit cubes that filled the unit cube?**

6. **Describe any patterns you noticed between the size of the fractional unit cube and the number of those cubes that fill the unit cube.**

Consider the formulas for volume of a prism and area of a rectangle:

$$V = l \cdot w \cdot h$$
$$A = l \cdot w$$

If you use B to represent the area of the base of a rectangular prism, then you can rewrite the formula for area: $B = l \cdot w$.

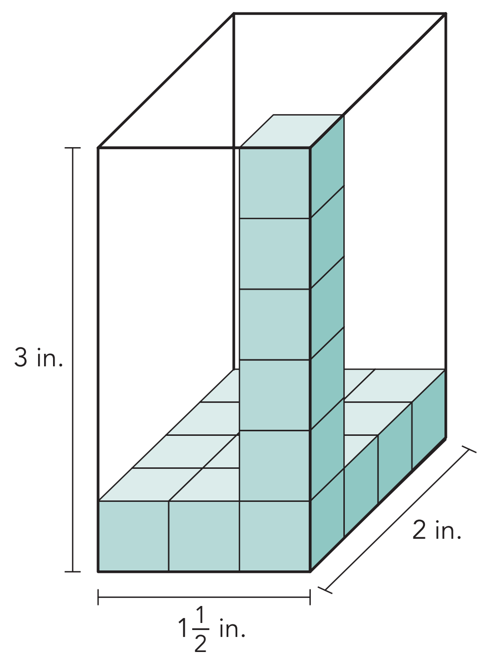
Using both of these formulas, you can rewrite the formula for the volume of a rectangular prism as $V = B \cdot h$, where V represents the volume, B represents the area of the base, and h represents the height.

ACTIVITY
4.2 Packing a Prism with Fractional Unit Cubes

Think about packing a rectangular prism with fractional unit cubes to determine its volume.

WORKED EXAMPLE

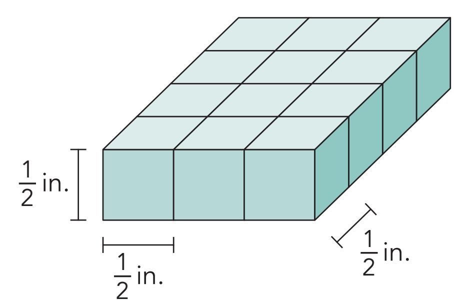
Consider the cube with dimensions $1\frac{1}{2}$ in. × 2 in. × 3 in.



Step 1: Pack the base of the prism with $\frac{1}{2}$-inch cubes.

Step 2: Analyze the base layer of fractional unit cubes that pack the prism.

There are 3 × 4, or 12, cubes that each have a volume of $\frac{1}{8}$ cubic inch in the base.



Step 3: Determine the number of fractional unit cubes that pack the prism.

There are 6 layers of cubes that make up the height of the prism, so there are 6 × 12, or 72, cubes that each have a volume of $\frac{1}{8}$ cubic inch in the prism.

Step 4: Determine the volume of the prism.

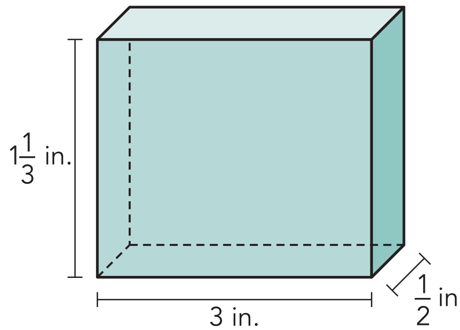
$72 \times \frac{1}{8}$ cubic inch
= 9 cubic inches

Step 5: Use the volume formula to check your answer.

$$V = 1\frac{1}{2} \cdot 2 \cdot 3$$
$$= \frac{3}{2} \cdot 2 \cdot 3$$
$$= 9 \text{ cubic inches}$$

1. Does it matter what size fractional unit cube you use to pack the right rectangular prism in the Worked Example? Give examples to explain your reasoning.

Consider packing the rectangular prism shown with fractional unit cubes to determine its volume. How can you determine which size cube to use?



WORKED EXAMPLE

Determine the least common multiple (LCM) of the fraction denominators to determine the dimensions of each cube.	LCM(2, 3) $= 6$ So, each cube will measure $\frac{1}{6}$ in. $\times$ $\frac{1}{6}$ in. $\times$ $\frac{1}{6}$ in. The volume of each unit cube is $\frac{1}{216}$ cubic inches.
Determine the number of unit cubes needed to pack the prism in each dimension.	length: $3 \div \frac{1}{6} = 18$ width: $\frac{1}{2} \div \frac{1}{6} = 3$ height: $1\frac{1}{3} \div \frac{1}{6} = 8$
Determine the total number of unit cubes that make up the right rectangular prism.	$(18)(3)(8) = 432$
Multiply the total number of unit cubes by the volume of each cube to determine the volume of the right rectangular prism.	$432\left(\frac{1}{216}\right) = 2$

The volume of the right rectangular prism is 2 cubic inches.

2. Interpret the Worked Example.

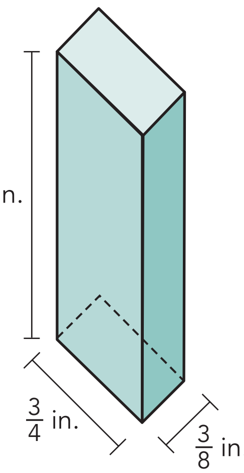
a. How was the number of unit cubes needed to pack the prism in each dimension determined?

b. Instead of $\frac{1}{6}$-inch cubes, suppose you used $\frac{1}{12}$-inch cubes. How does this change the volume of the rectangular prism?

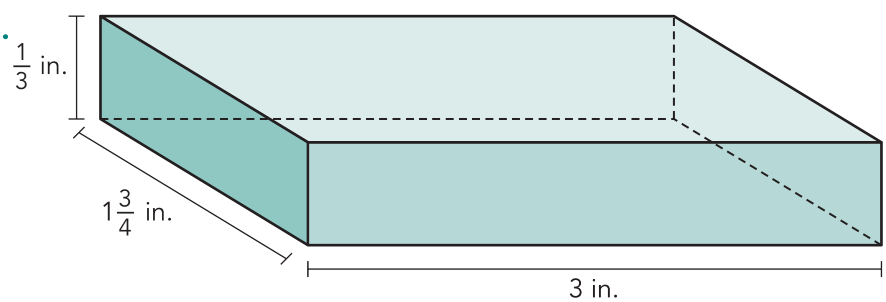
c. What is another way of determining the volume of the prism without packing it with cubes?

3. Use the method from the Worked Example to determine the volume of each rectangular prism. Then, use the volume formula to check your answer.

a.



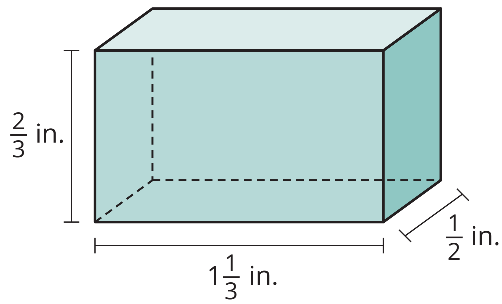
b.



ACTIVITY 4.3 Solving Volume Problems

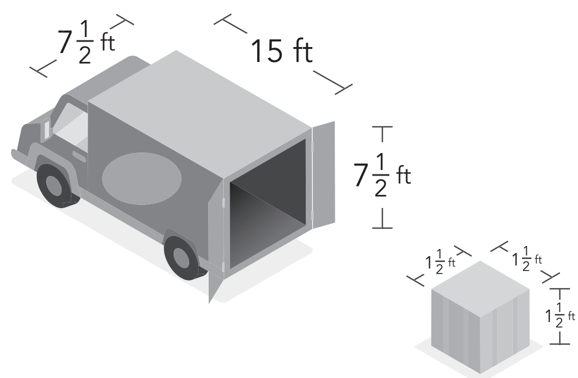
Use the unit fraction method to solve each volume problem.

1. **Calculate the volume of the right rectangular prism.**



2. **Arlene packs a moving truck with cube-shaped boxes that have side lengths of $1\frac{1}{2}$ feet. The back of the truck is a rectangular prism with the dimensions $7\frac{1}{2}$ feet by 15 feet by $7\frac{1}{2}$ feet.**

   a. **What is the volume of each box?**



   b. **Determine the number of boxes that will completely fill the back of the truck.**

   c. **Calculate the volume of the back of the moving truck.**

NOTES

TALK the TALK

Fractionally Full

Solve each problem. Show your work.

1. **Determine the volume of a right rectangular prism with dimensions $1\frac{1}{4}$ feet × 1 foot × $\frac{1}{2}$ foot using the unit fraction method you learned in this lesson.**

2. **Haley makes earrings and packages them in cubic boxes that measure $\frac{1}{6}$ foot wide. How many $\frac{1}{6}$-foot cubic boxes can she fit into a shipping box that is $1\frac{1}{6}$ feet by $\frac{1}{3}$ foot by $\frac{1}{3}$ foot?**

3. **The school athletic director has a storage closet that is $4\frac{1}{2}$ feet long, $2\frac{2}{3}$ feet deep, and 6 feet tall.**

   a. **She wants to put carpet in the closet. How much carpeting will she need?**

   b. **The athletic director wants to store cube-shaped boxes that are $\frac{1}{2}$ foot wide. How many boxes will the storage closet hold?**