Pulling a One-Eighty!

2

Triangle Sum and Exterior Angle Theorems

MATERIALS

Patty paper
Centimeter ruler

Lesson Overview

Students explore and justify the relationships between angles and sides in a triangle. They establish the Triangle Sum Theorem and use the theorem as they explore the relationship between interior angle measures and the side lengths of triangles. Students identify exterior angles and remote interior angles of triangles and explore the relationship between these angles to establish the Exterior Angle Theorem. They then practice applying both theorems to demonstrate their knowledge of triangle relationships.

Grade 7
Expressions, Equations, and Relationships

(11) The student applies mathematical process standards to solve one-variable equations and inequalities. The student is expected to:

(C) write and solve equations using geometry concepts, including the sum of the angles in a triangle, and angle relationships.

Grade 8
Expressions, Equations, and Relationships

(8) The student applies mathematical process standards to use one-variable equations or inequalities in problem situations. The student is expected to:

(D) use informal arguments to establish facts about the angle sum and exterior angle of triangles, the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 5.B, 5.F, 5.G

Essential Ideas

- The Triangle Sum Theorem states that the sum of the measures of the interior angles of a triangle is 180°.
- The longest side of a triangle lies opposite the largest interior angle.
- The shortest side of a triangle lies opposite the smallest interior angle.
- The remote interior angles of a triangle are the two angles non-adjacent to the exterior angle.
- The Exterior Angle Theorem states that the measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles of the triangle.

Lesson Structure and Pacing: 2 Days

Day 1

Engage

Getting Started: Rip 'Em Up

Students draw a large triangle on a piece of paper, rip off the three interior angles of the triangle, and fit them together in a puzzle-like fashion to form a line. This investigation informally justifies the Triangle Sum Theorem, which will be stated at the beginning of Activity 1.1.

Develop

Activity 2.1: Analyzing Angles and Sides

Students use the Triangle Sum Theorem to analyze the relationship between the lengths of the sides of a triangle and the measures of their opposite angles. They determine the unknown angle measure in three triangles and measure the sides of these triangles. Students notice the measure of an interior angle in a triangle is directly related to the side length opposite that angle.

Day 2

Activity 2.2: Exterior Angle Theorem

Students prove the Exterior Angle Theorem, which states that the measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles of the triangle. They use this theorem to algebraically solve for unknown angle measurements.

Demonstrate

Talk the Talk: So Many Angles!

Students demonstrate their knowledge of the triangle relationships learned in the lesson—the Triangle Sum Theorem and the Exterior Angle Theorem. The diagrams are complex and require students to use sides and angles as elements of different triangles.

ENGAGE

Getting Started: Rip 'Em Up

Facilitation Notes

In this activity, students use patty paper to informally justify the Triangle Sum Theorem.

Provide students with patty paper for this activity.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Differentiation strategy

To assist all students,

- Have students shade or label the vertices to make connecting the angles more explicit.
- Have students cut out the triangles from the patty paper so that manipulating the angles is easier.

Questions to ask

- What are adjacent angles?
- Do adjacent angles share a common side?
- Do adjacent angles share a common vertex?
- What kind of triangle did you draw?
- Is your triangle an acute, obtuse, or right triangle?
- Is your triangle scalene, isosceles, or equilateral?
- When positioned adjacent to each other, do the three angles form a straight line?
- What is a straight angle?
- What is the measure of a straight angle?
- How many degrees are associated with a line?
- Is there another way to arrange your three angles? Do you get the same result?
- Is the sum of the three interior angles the same for everyone's triangle?

Summary

The sum of the measures of the three interior angles of a triangle is equal to 180°.

Activity 2.1
Analyzing Angles and Sides

DEVELOP

Facilitation Notes

In this activity, students use the Triangle Sum Theorem to determine the measure of a third interior angle of a triangle, given the measure of two interior angles. They also explore the connection between the measure of an interior angle in a triangle and the side length opposite that angle.

Ask a student to read the introduction and theorem. Discuss as a class.

Provide students with rulers. Have students work with a partner or in a group to complete Questions 1 through 4. Share responses as a class.

Differentiation strategies

To assist all students,

- Place this lesson in context with their prior knowledge of triangles. Review the Triangle Inequality Theorem that deals with the relationship among the side lengths of a triangle. Explain that in this lesson, students will be dealing with the relationship among the angle measures and the relationship connecting angle measures and side measures.
- Explain how to label angles and sides of triangles. Have students label the vertex of each angle with a capital letter and the side opposite of an angle with the corresponding lowercase letter. When students list the angle and side measures, have them record the label and number. This will assist students in recognizing and explaining the relationship between angles and the sides of a triangle.

Questions to ask

- Is this triangle an acute, obtuse, or right triangle?
- Is this triangle scalene, isosceles, or equilateral?
- How can you determine the unknown angle measure in a triangle?
- How can the Triangle Sum Theorem be helpful in this situation?
- What happens to the length of a side opposite an interior angle of a triangle as the measure of the angle increases?
- What happens to the length of a side opposite an interior angle of a triangle as the measure of the angle decreases?

- What happens to the measure of the angle opposite a side of a triangle as the length of the side increases?
- What happens to the measure of the angle opposite a side of a triangle as the length of the side decreases?
- What side is always the longest side in an obtuse triangle? Explain.
- If two interior angles of a triangle are the same measure, how do the length of the sides opposite these angles compare to each other?
- Which side is always the longest side in a right triangle? Explain.

Have students work with a partner or in a group to complete Questions 5 and 6. Share responses as a class.

Questions to ask

- What is the measure of the third angle of the triangle?
- How does the unknown interior angle measure compare to the measure of the other two interior angles?
- Which side lies opposite the largest interior angle of the triangle?
- Which side lies opposite the smallest interior angle of the triangle?
- How do you determine the longest side of the triangle?
- How do you determine the shortest side of the triangle?
- What situation could result in no one side being the longest side of a triangle?
- What situation could result in no one side being the shortest side of a triangle?
- Why does this relationship between the sides and angles make sense?

Misconception

With regard to Question 6 part (c), students may think they can compare the sizes of all the angles in the figure without considering the angle measures within each triangle separately. If students consider only the angle measures, they might be unsure where to place *g* in the order of side lengths.

Summary

The Triangle Sum Theorem states that the sum of the measures of the interior angles of a triangle is 180°. The largest angle of a triangle lies opposite the longest side and the smallest angle of the triangle lies opposite the shortest side.

Activity 2.2
Exterior Angle Theorem

Facilitation Notes

In this activity, students informally prove the Exterior Angle Theorem. The theorem is used to algebraically solve for unknown angle measurements.

Have a student read the introduction and definition, and then complete Question 1 as a class.

Have students work with a partner or in a group to complete Questions 2 through 5. Share responses as a class.

Differentiation strategies

- Prove the Exterior Angle Theorem informally using patty paper. Have students trace the diagram, tear off Angles 1 and 2, and place them over Angle 4 to demonstrate the equality.
- Make sense of the Exterior Angle Theorem using numeric examples. Have students enter values for Angles 1 and 2, and then calculate the measures of Angles 3 and 4.
- Discuss the differences between the proof in the text, the informal patty paper proof, and using examples.

Misconceptions

- Students may think that only one exterior angle can be formed at each vertex. Demonstrate that there are two possible exterior angles at each vertex and discuss why they are congruent.
- Students are familiar with the term *remote* being associated with a remote control or a remote location. Make the connection between the mathematical definition of *remote* and its common use.

Questions to ask

- Where are the interior angles of a triangle located?
- Which angles are the interior angles in this triangle?
- Where is an exterior angle of a triangle located?
- Which angle is an exterior angle of this triangle?
- Is $\angle 4$ considered an interior angle of the triangle? Why or why not?
- What do you know about the relationship between $\angle 3$ and $\angle 4$?
- Are $\angle 3$ and $\angle 4$ a pair of adjacent angles?
- What are supplementary angles?
- Are $\angle 3$ and $\angle 4$ a pair of supplementary angles?

- What is a linear pair of angles?
- Are ∠3 and ∠4 a linear pair of angles?
- Are ∠1 and ∠2 adjacent or non-adjacent angles?
- Which side did you extend to create a second exterior angle?
- The second exterior angle is adjacent to which interior angle?

Ask a student to read the information and theorem following Question 5. Discuss as a class.

Have students work with a partner or in a group to complete Question 6. Share responses as a class.

Questions to ask

- What are the names of the interior angles?
- What is the name of the exterior angle?
- Considering the location of the exterior angle, which two interior angles are considered remote?
- What is the measure of $\angle ACB$? What theorem was helpful?
- What is the measure of $\angle ACD$? What theorem was helpful?
- If the measure of $\angle A$ is 16°, can you determine the measure of $\angle B$?
- If the measure of $\angle ACD$ is 143°, can you determine the measure of $\angle ACB$?
- If the measure of $\angle B$ is 31°, how does that help you determine the measure of $\angle A$?
- Which two angles are considered the remote interior angles in this situation?
- Which angle is the exterior angle in this situation?
- Do you know the measure of the two remote interior angles?
- Do you know the measure of the exterior angle?

Have students work with a partner or in a group to complete Question 7. Share responses as a class.

Questions to ask

- How would you describe the location of Angle *x*?
- Is Angle *x* considered an interior angle or an exterior angle? Explain.
- What is the measure of the other remote interior angle?
- What is the relationship between the measure of an exterior angle of a triangle and the measure of the two remote interior angles?
- What does it mean if two interior angles are both labeled as Angle *x*?
- How does knowing that the two remote interior angles of a triangle are congruent help you determine their measures?

- What information can you conclude from knowing that the measure of the exterior angle is 152°?
- Can the expressions $3x$ and 48 be combined?
- How do you combine the expressions $3x$ and 48?
- What can the expression $3x + 48$ be set equal to?
- Can the expressions $(2x + 6)$ and 40 be combined?
- How do you combine the expressions $(2x + 6)$ and 40?
- How could these problems be solved using arithmetic?
- How can you check that your angle measures are correct?

Summary

The Exterior Angle Theorem states that the measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles of the triangle.

Talk the Talk: So Many Angles!

DEMONSTRATE

Facilitation Notes

In this activity, students use the Triangle Sum Theorem, the relationship between side lengths and angle measures within a triangle, and the Exterior Angle Theorem to determine unknown angle measurements.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Differentiation strategies

- To scaffold support,
  - Provide an enlarged diagram.
  - Suggest that students outline/color triangles different colors so that they can see the different triangles in each diagram.
  - Review that the sum of the angles around a point is 180°.
- To extend the activity, have students find an example where two small triangles can be combined to create one larger triangle. Ask students to identify the angle measures of each of the smaller triangles to calculate that the sum of the angle measures for each triangle is 180°. Have students use their measurements to verify that the sum of angle measures for the larger triangle is also 180°.

Misconception

Students may begin to overgeneralize the concept of exterior angles. For example, they may think that the 35° near the center of the diagram has an exterior angle drawn adjacent to it. Point out the fact that exterior angles always create a linear pair, and in this case there is not a straight line to create a linear pair.

Questions to ask

- Which angle did you determine first? What theorem did you use to determine the measure of this angle?
- Which angle did you determine next? What theorem did you use to determine the measure of this angle?
- Was the Triangle Sum Theorem helpful? Where?
- Was the Exterior Angle Theorem helpful? Where?

Summary

The Triangle Sum Theorem, the relationship between lengths of sides and angle measures within a triangle, and the Exterior Angle Theorem can be used to determine unknown angle measurements.

Pulling a One-Eighty!

Triangle Sum and Exterior Angle Theorems

2

Warm Up Answers

1. $x = 75$
2. $x = 57.5$
3. $x = 37.5$
4. $x = 45$

WARM UP

Solve each equation for x.

1. $x + 105 = 180$
2. $2x + 65 = 180$
3. $45 + 4x - 15 = 180$
4. $90 + 2x = 180$

LEARNING GOALS

- Establish the Triangle Sum Theorem.
- Explore the relationship between the interior angle measures and the side lengths of a triangle.
- Identify the remote interior angles of a triangle.
- Identify the exterior angles of a triangle.
- Use informal arguments to establish facts about exterior angles of triangles.
- Explore the relationship between the exterior angle measures and two remote interior angles of a triangle.
- Prove the Exterior Angle Theorem.

KEY TERMS

- Triangle Sum Theorem
- exterior angle of a polygon
- remote interior angles of a triangle
- Exterior Angle Theorem

You already know a lot about triangles. In previous grades, you classified triangles by side lengths and angle measures. What special relationships exist among the interior angles of a triangle and between interior and exterior angles of a triangle?

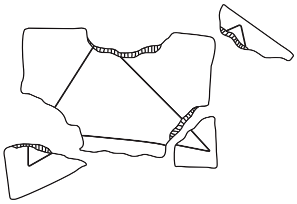
Answers

1. The three angles form a straight line, so the sum of the angles is 180°.
2. Answers may vary.

Getting Started

Rip 'Em Up

Draw any triangle on a piece of patty paper. Tear off the triangle's three angles. Arrange the angles so that they are adjacent angles.



1. What do you notice about these angles? Write a conjecture about the sum of the three angles in a triangle.

2. Compare your angles and your conjecture with your classmates'. What do you notice?

2 • TOPIC 3: Line and Angle Relationships

ELL Tip

To help English Language Learners as they complete Question 1 of Activity 1.1, a word bank could be provided that shows the different classification of triangles for students to choose. Have the word bank printed on a small sheet of paper that can be easily placed on students' desks.

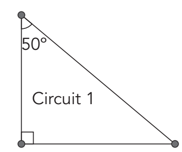
ACTIVITY 2.1

Analyzing Angles and Sides

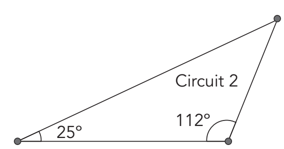
In the previous activity, what you noticed about the relationship between the three angles in a triangle is called The *Triangle Sum Theorem*. The **Triangle Sum Theorem** states that the sum of the measures of the interior angles of a triangle is 180°.

Trevor is organizing a bike race called the Tri-Cities Criterium. Criteriums consist of several laps around a closed circuit. Based on the city map provided to him, Trevor designs three different triangular circuits and presents scale drawings of them to the Tri-Cities Cycling Association for consideration.

1. Classify each circuit according to the type of triangle created.

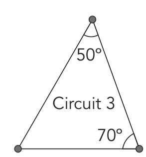


2. Use the Triangle Sum Theorem to determine the measure of the third angle in each triangular circuit. Label the triangles with the unknown angle measures.



3. Measure the length of each side of each triangular circuit. Label the side lengths in the diagram.

The sharper the angles on a race course, the more difficult the course is for cyclists to navigate.



4. Perform the following tasks for each circuit.

a. List the angle measures from least to greatest.

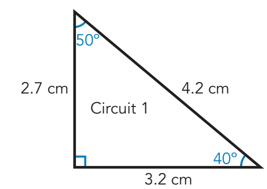
b. List the side lengths from shortest to longest.

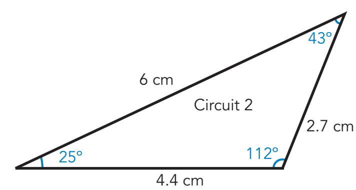
ELL Tip

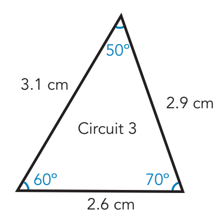
Have students make a flash card for the definition of the Triangle Sum Theorem. On one side of the card have students write "Triangle Sum Theorem," and on the opposite side have students write the theorem. Encourage students to write the definition in their own words, using pictures or symbols.

Answers

1. Circuit 1 is a right triangle. Circuit 2 is an obtuse triangle. Circuit 3 is an acute triangle.
2. The measure of the third angle in Circuit 1 is 40°. The measure of the third angle in Circuit 2 is 43°. The measure of the third angle in Circuit 3 is 60°.
3.







4a. Circuit 1: 40°, 50°, 90°

Circuit 2: 25°, 43°, 112°

Circuit 3: 50°, 60°, 70°

4b. Circuit 1: 2.7 cm, 3.2 cm, 4.2 cm

Circuit 2: 2.7 cm, 4.4 cm, 6 cm

Circuit 3: 2.6 cm, 2.9 cm, 3.1 cm

Answers

4c. The shortest side is always across from the angle with the least measure of the triangle.

4d. The longest side is always across from the angle with the greatest measure of the triangle.

5a. The third angle has a measure of 62°.

5b. The longest side of the triangle lies opposite the 62° angle.

5c. The shortest side of the triangle lies opposite the 57° angle.

c. **Describe what you notice about the location of the angle with the least measure and the location of the shortest side.**

d. **Describe what you notice about the location of the angle with the greatest measure and the location of the longest side.**

5. Traci, the president of the Tri-Cities Cycling Association, presents a fourth circuit for consideration. The measures of two of the interior angles of the triangle are 57° and 61°. Determine the measure of the third angle, and then describe the location of each side with respect to the measures of the opposite interior angles without drawing or measuring any part of the triangle.

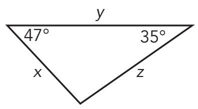
a. **measure of the third angle**

b. **longest side of the triangle**

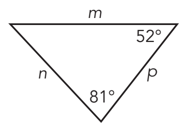
c. **shortest side of the triangle**

6. List the side lengths from shortest to longest for each diagram.

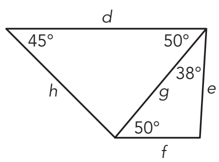
a.

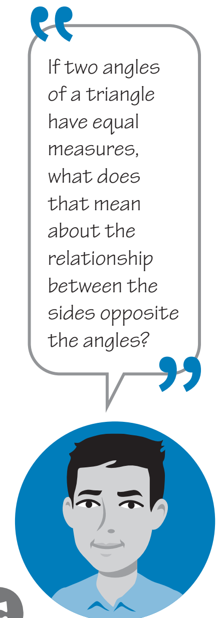


b.



c.



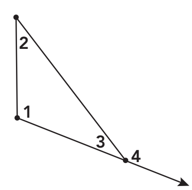


ACTIVITY 2.2

Exterior Angle Theorem

You now know about the relationships among the angles inside a triangle, the *interior angles of a triangle*, but are there special relationships between interior and *exterior angles of a triangle*?

An **exterior angle of a polygon** is an angle between a side of a polygon and the extension of its adjacent side. It is formed by extending a ray from one side of the polygon.



In the diagram, ∠1, ∠2, and ∠3 are the interior angles of the triangle, and ∠4 is an exterior angle of the triangle.

1. Make a conjecture about the measure of the exterior angle in relation to the measures of the other angles in the diagram.

Answers

6a. *x, z, y*

6b. *p, n, m*

6c. *f, e, g, h, d*

Answers

1. Conjectures will vary.

Answers

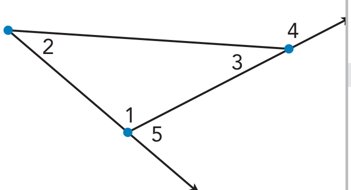
2a. $m\angle 1 + m\angle 2 + m\angle 3 = 180°$. The Triangle Sum Theorem states that the sum of the measures of the interior angles of triangle is equal to 180°.

2b. $m\angle 3 + m\angle 4 = 180°$. Angle 3 and Angle 4 form a linear pair of angles. Linear pairs are supplementary, so the sum of their measures is 180°.

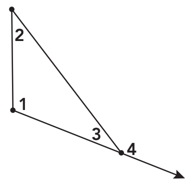
2c. $m\angle 1 + m\angle 2 = m\angle 4$. If $m\angle 1 + m\angle 2 + m\angle 3 = 180°$ and $m\angle 3 + m\angle 4 = 180°$, then $m\angle 1 + m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$, by substitution. Subtracting $m\angle 3$ from both sides of the equation results in $m\angle 1 + m\angle 2 = m\angle 4$.

3a. Considering all three interior angles of the triangle, $\angle 1$ and $\angle 2$ are the two interior angles that are farthest away from, or not adjacent to, $\angle 4$.

3b. Sample answer. Angle 2 and Angle 3 are the remote interior angles with respect to Angle 5.



4. The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles of the triangle.



2. Let's investigate the relationships among measures of the angles in the diagram.

a. What does $m\angle 1 + m\angle 2 + m\angle 3$ equal? Explain your reasoning.

b. What does $m\angle 3 + m\angle 4$ equal? Explain your reasoning.

c. State a relationship between the measures of $\angle 1$, $\angle 2$, and $\angle 4$. Explain your reasoning.

3. In a triangle, for each exterior angle there are two "remote" interior angles.

a. Why would $\angle 1$ and $\angle 2$ be referred to as "remote" interior angles with respect to the exterior angle, $\angle 4$?

b. Extend another side of the triangle and label the exterior angle $\angle 5$. Then name the two remote interior angles with respect to $\angle 5$.

The **remote interior angles of a triangle** are the two angles that are non-adjacent to the specified exterior angle.

4. Rewrite $m\angle 4 = m\angle 1 + m\angle 2$ using the terms *sum, remote interior angles of a triangle,* and *exterior angle of a triangle.*

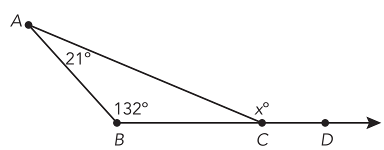
5. **The original diagram was drawn as an obtuse triangle with one exterior angle. If the triangle had been drawn as an acute or right triangle, would this have changed the relationship between the measure of the exterior angle and the sum of the measures of the two remote interior angles? Explain your reasoning.**

Was your conjecture from Question 1 correct? If so, you have proven an important theorem in the study of geometry!

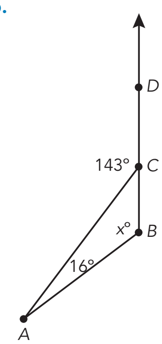
The **Exterior Angle Theorem** states that the measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles of the triangle.

6. **Use the Exterior Angle Theorem to determine each unknown angle measure.**

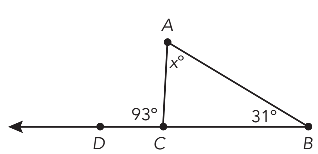
a.



b.



c.



Answers

5. No. The sum of the measures of the remote interior angles would still be equal to the measure of the exterior angle. The calculations in Question 2 were not based on the type of triangle.

6a. I can conclude that $m\angle ACD = 153°$.

6b. I can conclude that $m\angle ABC = 127°$.

6c. I can conclude that $m\angle BAC = 62°$.

Answers

7a. $108 + x = 156; x = 48$

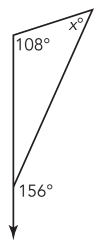
7b. $2x = 152; x = 76$

7c. $3x + 48 = 120; x = 24$

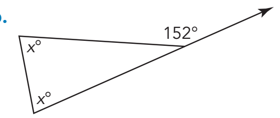
7d. $2x + 6 + 40 = 126;$
$x = 40$

7. Write and solve an equation to determine the value of *x* in each diagram.

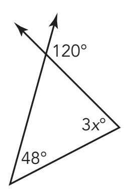
a.



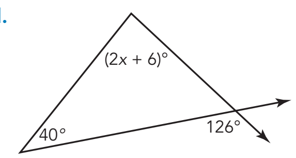
b.



c.



d.

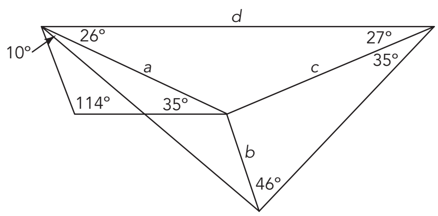


NOTES

TALK the TALK

So Many Angles!

1. Consider the diagram shown.



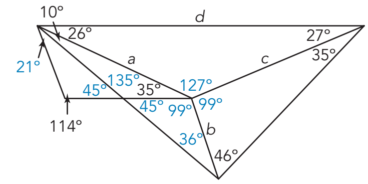
   a. Determine the measures of the eight unknown angle measures inside the figure.

   b. List the labeled side lengths in order from least to greatest.

LESSON 2: Pulling a One-Eighty! • 9

Answers

1a.



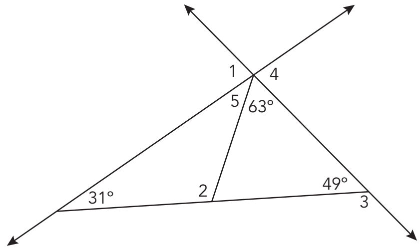
1b. *b*, *c*, *a*, *d*

Answers

2. $m\angle 1 = 80°$,
   $m\angle 2 = 112°$,
   $m\angle 3 = 131°$,
   $m\angle 4 = 80°$,
   $m\angle 5 = 37°$

NOTES

2. Determine the unknown angle measures in the figure.



10 • TOPIC 3: Line and Angle Relationships