# Tagging Sharks

3

Solving Proportions Using Means and Extremes

## MATERIALS

None

## Lesson Overview

Students solve several proportions embedded in real world contexts. The term *variable* is introduced to represent an unknown quantity. Several proportions that contain one variable are solved using one of three methods: the scaling method, the unit rate method, and the means and extremes method. Students learn to isolate a variable in a proportion by using inverse operations.

## Grade 7
## Proportionality

**(4) The student applies mathematical process standards to represent and solve problems involving proportional relationships. The student is expected to:**

(C) determine the constant of proportionality $\left(k = \frac{y}{x}\right)$ within mathematical and real-world problems.

(D) solve problems involving ratios, rates, and percents, including multi-step problems involving percent increase and percent decrease, and financial literacy problems.

## ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 4.D, 4.J, 5.B, 5.F, 5.G

## Essential Ideas

- A variable is a letter or symbol used to represent a number.
- To solve a proportion means to determine all the values of the variables that make the proportion true.
- A method for solving a proportion called the scaling method involves multiplying (scaling up) or dividing (scaling down) the numerator and denominator of one ratio by the same factor until the denominators of both ratios are the same number.
- A method for solving a proportion called the unit rate method involves changing one ratio to a unit rate and then scaling up to the rate you need.

- A method for solving a proportion called the means and extremes method involves identifying the means and extremes, and then setting the product of the means equal to the product of the extremes to solve for the unknown quantity.
- Isolating a variable involves performing an operation, or operations, to get the variable by itself on one side of the equals sign.
- Inverse operations are operations that undo each other such as multiplication and division, or addition and subtraction.

## Lesson Structure and Pacing: 2 Days

### Day 1

#### Engage

**Getting Started: Mix-N-Match**

Students recall the definition of *proportion* and analyze how the numerators and denominators of the ratios in a proportion can be rearranged to create other true proportions.

#### Develop

**Activity 3.1: Maintaining Equality with Proportions**

Students learn about how biologists tag sharks in order to study them and how scientists might use proportions to estimate shark populations based on tagged sharks. Students investigate different scenarios and represent these situations using proportions. They then write a proportion with variables in several different ways while maintaining equality.

### Day 2

**Activity 3.2: Solving Proportions with Means and Extremes**

Students investigate a scenario in order to solve a proportion with an unknown. Students are exposed to several different strategies for solving a proportion, including setting the products of the means and extremes equal to each other. Students use inverse operations in order to isolate the variable and solve a proportion for the variable.

**Activity 3.3: Solving Problems with Proportions**

Students solve a variety of real-world problems by setting up and solving proportions. Students also use a table to represent a proportional relationship in this activity.

#### Demonstrate

**Talk the Talk: Choose Your Own Proportion Adventure**

Students are given proportions, including one with a complex ratio, and are asked to write a scenario that each proportion could represent. They then solve the proportions and show their solutions.

ENGAGE

# Getting Started: Mix-N-Match

## Facilitation Notes

In this activity, students review the definition of *proportion*. An example rearranges the numerators and denominators to create other true proportions.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

**Differentiation strategies**

To scaffold support,

- Provide a context, such as, in a daycare center there are 2 adults for every 4 infants.
- Have students enter the units next to each value. For example,
  $\frac{\text{adults}}{\text{ADULTS}} = \frac{\text{infants}}{\text{INFANTS}}$ or $\frac{\text{adults}}{\text{infants}} = \frac{\text{ADULTS}}{\text{INFANTS}}$ or $\frac{\text{infants}}{\text{adults}} = \frac{\text{INFANTS}}{\text{ADULTS}}$
- After students provide the units, go back and demonstrate how lowercase and capital letters can be used to distinguish between the two pairs of numbers.
- Have students explain how corresponding values are aligned in the proportions.
- Have students create a new context for Example 2 and repeat the process.

**Questions to ask**

- Examine the equations in the table. How can you verify that all the equations are true?
- Which numerator remained in the same position? Which numerator changed positions with a denominator?
- Which denominator remained in the same position? Which denominator changed positions with a numerator?

## Summary

A proportion is an equation that states that two ratios are equal.

# Activity 3.1
## Maintaining Equality with Proportions

DEVELOP

### Facilitation Notes

In this activity, students write a proportion with variables in different ways while maintaining equality. Proportions are used to estimate populations.

Ask a student to read the introduction and discuss as a class. Have students work with a partner or in groups to complete Questions 1 through 3. Share responses as a class.

**Differentiation strategies**

Complete a capture-recapture experiment as a class.

- Have a student read the information at the top of the page. Complete active reading strategies as a class, numbering and discussing the steps.
- Continue the process, having students read part of the text, then stop and check for understanding.
- Reenact the process using a large box filled with multiple identical items, such as marbles, chips or cubes. Present it to the class and have all students guess how many of the items are in the box.
- Walk through the numbered steps, paralleling the process with the item you chose to use. As a way of tagging the items, write on them with marker or replace them with the same item in a different color.
- Complete the recapture method several times and take an average.
- Compare the estimated total the class calculated with students' guesses and the actual total.

**Questions to ask**

- How did you know what quantity to write in the numerators of each ratio?
- How did you know what quantity to write in the denominators of each ratio?
- How many quantities in the proportion are unknown or represented by a variable?
- Which quantity in the proportion is represented by a variable?
- Is it possible to solve a proportion if it contained two variables or unknown quantities?

Have students work with a partner to complete Question 4. Share responses as a class.

**Questions to ask**

- What proportion did you use to solve for the number of deer in the refuge?
- How did you solve this proportion?
- How can you be sure your new proportion is equal to the given proportion?
- Did you follow any rule when you wrote the proportion differently?
- In how many different ways can one proportion be rewritten?

Have students work with a partner or in groups to complete Questions 5 and 6. Share responses as a class.

**Questions to ask**

- When you rearranged the variables, how did you decide if it was equal to the original proportion?
- What products did you maintain in the numerator and denominator positions to ensure equity when you set up a different proportion?

## Summary

Proportions can be rewritten different ways as long as equality is maintained.

# Activity 3.2

## Solving Proportions with Means and Extremes

## Facilitation Notes

In this activity, students investigate a scenario in order to solve a proportion with an unknown. Students are exposed to several different strategies for solving a proportion, including setting the products of the means and the products of the extremes equal to each other. Students use inverse operations in order to isolate the variable and solve a proportion for the variable.

Ask a student to read the introduction aloud. Review the different methods and complete Questions 1 through 3 as a class.

**Questions to ask**

- How are John David's and Parker's methods different?
- How did you verify Natalie's method?

- Does using all three methods to solve this proportion always give you the same answer?
- Is one method easier to use than the other methods?
- How are the three methods similar to each other?
- How are the three methods different from each other?

Have students work with a partner or in a group to complete Question 4. Share responses as a class.

**Questions to ask**

- What method did you use to solve each proportion?
- Did the position of the unknown quantity factor into your method for solving the proportion?
- How can you check your solution to make sure it is correct?

Ask a student to read the information, examples, and definitions following Question 4 aloud. Discuss as a class.

Have students work with a partner or in a group to complete Questions 5 through 6. Share responses as a class.

**Questions to ask**

- What kind of action would cause an equation not to maintain its balance?
- What is the product of the means in this proportion?
- What is the product of the extremes in this proportion?
- What is the value of the left side of the equation?
- What is the value of the right side of the equation?
- Do both sides of each equation have the same value?
- How was balance maintained?

Ask a student to read the information, examples, and definitions following Question 6 aloud. Discuss as a class.

Have students work with a partner or in a group to complete Questions 7 and 8. Share responses as a class.

**Questions to ask**

- How many variables are in this proportion?
- Are $a$, $b$, $c$, and $d$ all unknown quantities?
- Why is it important to learn how to isolate each variable?
- Is there more than one way to isolate a variable?

**Differentiation strategy**

To extend the activity, use multiplication similar to the second Worked Example to prove why the product of the means equals the product of the extremes.

$$\frac{a}{b} = \frac{c}{d}$$

$$(bd)\,\frac{a}{b} = \frac{c}{d}\,(bd) \quad \text{Multiply both sides by } bd.$$

$$ad = bc$$

### Summary

A formal method used to solve a proportion is to set the product of the means equal to the product of the extremes and isolate the variable.

## Activity 3.3
### Solving Problems with Proportions

### Facilitation Notes

In this activity, students use any method to solve problems using proportions.

Have students work with a partner or in a group to complete Questions 1 through 4. Share responses as a class.

**Differentiation strategy**

Have groups of 4 students complete the problems. One pair completes the even-numbered problems first, and the other pair completes the odd-numbered problems first. Pairs switch their work. The second pair to complete the problems must write the proportions a different way to solve the problems.

**Questions to ask**

- What proportion did you use to compute the weight of a person on Earth?
- What is the unit rate of water flowing over Niagara Falls every minute?
- How did you calculate the value of $1.00 in euros?
- How did you calculate the value of 6 euros in dollars?
- What proportions did you use to determine how much of each of the ingredients is needed?
- How did you decide which quantity to place in the numerator?
- How did you decide which quantity to place in the denominator?
- Did you use the same method to solve each proportion?

### Summary

A formal method used to solve a proportion is to set the product of the means equal to the product of the extremes and isolate the variable.

## Talk the Talk: Choose Your Own Proportion Adventure

DEMONSTRATE

### Facilitation Notes

In this activity, students craft problem situations to fit given proportions.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

**Differentiation strategy**

To scaffold support, suggest that students first decide if they are going to use a horizontal relationship or vertical relationship between the 2 quantities. Then, using two known quantities, they can create a problem situation.

**Questions to ask**

- What unit describes the quantities in your situation?
- What is the unit of measurement in your situation?
- Is there another way to write this proportion?
- What is another way to write this proportion?
- Does more than one value make this proportion true?
- Is there more than one correct answer in this situation?

### Summary

Proportions are used to represent real-world problem situations.

NOTES

# Tagging Sharks

# 3

## Solving Proportions Using Means and Extremes

### WARM UP

Solve each equation.

1. $w - 5 = 25$
2. $9x = 990$
3. $\frac{c}{12} = 48$
4. $1.15 + m = 10$

### LEARNING GOALS

- Rewrite proportions to maintain equality.
- Represent proportional relationships by equations.
- Develop strategies to solve proportions.
- Use proportional relationships to solve multistep problems.

### KEY TERMS

- proportion
- variable
- means
- extremes
- solve a proportion
- isolate the variable
- inverse operations

You have learned how to write proportions and calculate unknown values through scaling up and scaling down. Is there a more efficient strategy that works for any unknown in any proportion?



### Warm Up Answers

1. $w = 30$
2. $x = 110$
3. $c = 576$
4. $m = 8.85$

## Answers

| | Example 1 | Example 2 |
|---|---|---|
| 1. | $\frac{2}{3} = \frac{4}{6}$ | $\frac{5}{7} = \frac{15}{21}$ |
| 1a. | $\frac{6}{3} = \frac{4}{2}$ | $\frac{21}{7} = \frac{15}{5}$ |
| 1b. | $\frac{2}{4} = \frac{3}{6}$ | $\frac{5}{15} = \frac{7}{21}$ |

# Getting Started

## Mix-N-Match

Recall that a **proportion** is an equation that states that two ratios are equal.

A proportion can be written several ways. Each example shows three proportions using the same four quantities.

| | Example 1 | Example 2 |
|---|---|---|
| Proportion 1 | $\frac{2}{3} = \frac{4}{6}$ | $\frac{5}{7} = \frac{15}{21}$ |
| Proportion 2 | $\frac{6}{3} = \frac{4}{2}$ | $\frac{21}{7} = \frac{15}{5}$ |
| Proportion 3 | $\frac{2}{4} = \frac{3}{6}$ | $\frac{5}{15} = \frac{7}{21}$ |

1. **In each example, use arrows to show how the numbers were rearranged from the:**

   a. **first proportion to the second proportion.**

   b. **first proportion to the third proportion.**

ACTIVITY 3.1

## Maintaining Equality with Proportions

Because it is impossible to count each individual animal, marine biologists use a method called the capture-recapture method to estimate the population of certain sea creatures. In certain areas of the world, biologists randomly catch and tag a given number of sharks. After a period of time, such as a month, they recapture a second sample of sharks and count the total number of sharks as well as the number of recaptured tagged sharks. Then, the biologists use proportions to estimate the population of sharks living in a certain area.

Biologists can set up a proportion to estimate the total number of sharks in an area.

$$\frac{\text{Original number of tagged sharks}}{\text{Total number of sharks in an area}} = \frac{\text{Number of recaptured tagged sharks}}{\text{Number of sharks caught in the second sample}}$$

Although capturing the sharks once is necessary for tagging, it is not necessary to recapture the sharks each time. At times, the tags can be observed through binoculars from a boat or at shore.

Biologists originally caught and tagged 24 sharks off the coast of Cape Cod, Massachusetts, and then released them back into the bay. The next month, they caught 80 sharks with 8 of the sharks already tagged. To estimate the shark population off the Cape Cod coast, biologists set up the following proportion:

$$\frac{24 \text{ tagged sharks}}{p \text{ total sharks}} = \frac{8 \text{ recaptured tagged sharks}}{80 \text{ total sharks}}$$

Notice the variable $p$ in the proportion. In this proportion, let $p$ represent the total shark population off the coast of Cape Cod.

A **variable** is a letter or symbol used to represent a number.

## Answers

1. $\frac{24 \text{ tagged sharks}}{8 \text{ observed tags}} = \frac{p \text{ total sharks}}{80 \text{ sharks in second sample}}$

   $\frac{p \text{ total sharks}}{24 \text{ tagged sharks}} = \frac{80 \text{ sharks in second sample}}{8 \text{ observed tags}}$

   $\frac{8 \text{ observed tags}}{24 \text{ tagged sharks}} = \frac{80 \text{ sharks in second sample}}{p \text{ total sharks}}$

2. There are approximately 240 sharks off the coast of Cape Cod.
3. Answers will vary.
4. $\frac{240}{d} = \frac{30}{180}$; approximately 1440 deer in the refuge

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1. Write three additional different proportions you could use to determine the total shark population off the coast of Cape Cod.

2. Estimate the total shark population using any of the proportions.

3. Did any of the proportions seem more efficient than the other proportions?

4. Wildlife biologists tag deer in wildlife refuges. They originally tagged 240 deer and released them back into the refuge. The next month, they observed 180 deer of which 30 deer were tagged. Approximately how many deer are in the refuge? Write a proportion and show your work to determine your answer.

4 • TOPIC 2: Fractional Rates

A proportion of the form $\frac{a}{b} = \frac{c}{d}$ can be written in many different ways.

Another example is $\frac{d}{b} = \frac{c}{a}$ or $\frac{c}{a} = \frac{d}{b}$.

**5. Show how the variables were rearranged from the proportion in the "if" statement to each proportion in the "then" statement to maintain equality.**

**If $\frac{a}{b} = \frac{c}{d}$ then $\frac{d}{b} = \frac{c}{a}$.**

**If $\frac{d}{b} = \frac{c}{a}$ then $\frac{c}{a} = \frac{d}{b}$.**

**6. Write all the different ways you can rewrite the proportion $\frac{a}{b} = \frac{c}{d}$ and maintain equality.**

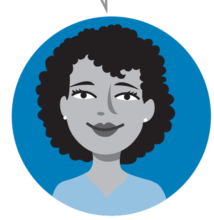
ACTIVITY 3.2

## Solving Proportions with Means and Extremes

The Ready Steady Battery Company tests batteries as they come through the assembly line and then uses a proportion to predict how many batteries in its total production might be defective.

On Friday, the quality controller tested every tenth battery and found that of the 320 batteries tested, 8 were defective. If the company shipped a total of 3200 batteries, how many might be defective?

"A quality control department checks the product a company creates to ensure that the product is not defective."





## Answers

5. $\frac{d}{b} \quad \frac{c}{a}$

$\frac{c}{a} \quad \frac{d}{b}$.

6. $\frac{a}{b} = \frac{c}{d}$ or $\frac{c}{d} = \frac{a}{b}$

$\frac{d}{b} = \frac{c}{a}$ or $\frac{c}{a} = \frac{d}{b}$

$\frac{d}{c} = \frac{b}{a}$ or $\frac{b}{a} = \frac{d}{c}$

$\frac{a}{c} = \frac{b}{d}$ or $\frac{b}{d} = \frac{a}{c}$

## Answers

1. Both Parker and John David scaled up the ratios to determine the number of defective batteries.

NOTES

Let's analyze a few methods.

**John David**

$$\frac{8 \text{ defective batteries}}{320 \text{ batteries}} = \frac{d \text{ defective batteries}}{3200 \text{ batteries}}$$

$$\frac{8}{320} \overset{\times 10}{=} \frac{d}{3200}$$

$$d = 80$$

So, 80 batteries might be defective.

**Parker**

8 defective batteries : 320 batteries

$\times 10$ $\times 10$

d defective batteries : 3200 batteries

$d = 80$

About 80 batteries will probably be defective.

**1. How are Parker's and John David's methods similar?**

NOTES

## Nora

$$\frac{8 \text{ defective batteries}}{320 \text{ batteries}} \overset{\div 8}{=} \frac{1 \text{ defective battery}}{40 \text{ batteries}} \overset{\times 80}{=} \frac{80 \text{ defective batteries}}{3200 \text{ batteries}}$$

One out of every 40 batteries is defective. So, out of 3200 batteries, 80 batteries could be defective because 3200 ÷ 40 = 80.

**2. Describe the strategy Nora used.**

## Natalie

When I write Nora's ratios using colons like Parker, I notice something about proportions . . .

8 : 320 = 1 : 40

8 : 1
320 : 40

1 : 40 = 80 : 3200

$\frac{1}{40} = \frac{80}{3200}$

. . . the two middle numbers have the same product as the two outside numbers. So, I can solve any proportion by setting these two products equal to each other.

**3. Verify that Natalie is correct.**



## Answers

2. First, Nora scaled down the original ratio. Then, she scaled up that ratio to determine the total defective batteries out of 3200 total batteries.
3. Natalie's solution: (8)(40) = 320 and (320)(1) = 320

## Answers

4a. Explanations will vary.
$g = 1$

4b. Explanations will vary.
$q = 45$

4c. Explanations will vary.
$d = 6$

**4. Try the various proportion-solving methods on these proportions and determine the unknown value. Explain which method you used.**

**a.** $\frac{\text{3 granola bars}}{\text{420 calories}} = \frac{g\text{ granola bars}}{\text{140 calories}}$

**b.** **8 correct : 15 questions = 24 correct : *q* questions**

**c.** $\frac{d\text{ dollars}}{\text{5 miles}} = \frac{\$9}{\text{7.5 miles}}$


The relationship that Natalie noticed is between the *means* and *extremes*. In a proportion that is written a : b = c : d, the product of the two values in the middle (the **means**) equals the product of the two values on the outside (the **extremes**).

extremes

$a : b = c : d$

means

$bc = ad$

or

$\frac{a}{b} = \frac{c}{d}$

means extremes

$bc = ad$

when $b \neq 0$, $d \neq 0$

To solve a proportion using this method, first identify the means and extremes. Then, set the product of the means equal to the product of the extremes and solve for the unknown quantity. To **solve a proportion** means to determine all the values of the variables that make the proportion true.

**WORKED EXAMPLE**

You can rewrite a proportion as the product of the means equal to the product of the extremes.

7 books : 14 days = 3 books : 6 days

means

extremes

$$(14)(3) = (7)(6)$$

$$42 = 42$$

$$\frac{7 \text{ books}}{14 \text{ days}} = \frac{3 \text{ books}}{6 \text{ days}}$$

$$(14)(3) = (7)(6)$$

$$42 = 42$$

**5. You can rewrite the product of the means and extremes from the Worked Example as four different equations. Analyze each equation.**

$$3 = \frac{(7)(6)}{14} \qquad 14 = \frac{(7)(6)}{3} \qquad \frac{(3)(14)}{7} = 6 \qquad \frac{(3)(14)}{6} = 7$$

**a. Why are these equations all true? Explain your reasoning.**

**b. Compare these equations to the equation in the Worked Example showing the product of the means equal to the product of the extremes. How was the balance of the equation maintained in each?**

**6. Why is it important to maintain balance in equations?**



## Answers

5a. Both sides of each equation have the same value.

5b. The same operation was performed on each side of the equation to maintain the balance.

6. For equations to be true, both sides must represent the same value. So, we must keep equations balanced in order to keep the equations true.

## Answers

7. Step 1 shows that the products of the means and of the extremes are equal. The goal was to isolate the variable *a*, so both sides were divided by *d* in Step 2. The proportion was simplified in Step 3.
8. See answers below.

NOTES

**WORKED EXAMPLE**

In the proportion $\frac{a}{b} = \frac{c}{d}$ you can multiply both sides by *b* to isolate the variable *a*.

$$b \cdot \frac{a}{b} = b \cdot \frac{c}{d} \longrightarrow a = \frac{bc}{d}$$

When you **isolate the variable** in an equation, you perform an operation, or operations, to get the variable by itself on one side of the equals sign. Multiplication and division are *inverse operations*. **Inverse operations** are operations that "undo" each other.

**WORKED EXAMPLE**

Another strategy to isolate the variable *a* is to multiply the means and extremes and then isolate the variable by performing inverse operations.

$$\frac{a}{b} = \frac{c}{d}$$

**Step 1:** $ad = bc$

**Step 2:** $\frac{ad}{d} = \frac{bc}{d}$

**Step 3:** $a = \frac{bc}{d}$

**7. Describe each step shown.**

**8. Rewrite the proportion $\frac{a}{b} = \frac{c}{d}$ to isolate each of the other variables: *b, c,* and *d*. Explain the strategies you used to isolate each variable.**



8. $b = \frac{ad}{c}$ I multiplied the means and extremes, and then divided both sides by *c*.

$c = \frac{ad}{b}$ I multiplied the means and extremes, and then divided both sides by *b*.

$d = \frac{cb}{a}$ I multiplied the means and extremes, and then divided both sides by *a*.

ACTIVITY
3.3

## Solving Problems with Proportions

Write and solve proportions to solve each problem.

1. **An astronaut who weighs 85 kilograms on Earth weighs 14.2 kilograms on the Moon. How much would a person weigh on the Moon if they weigh 95 kilograms on Earth? Round your answer to the nearest tenth.**

2. **Water goes over Niagara Falls at a rate of 180 million cubic feet every $\frac{1}{2}$ hour. How much water goes over the falls in 1 minute?**

3. **The value of the U.S. dollar in comparison to the value of foreign currency changes daily. Complete the table shown. Round to the nearest hundredth.**

| Euro | U.S. Dollar |
|---|---|
| 1 | 1.07 |
| | 1.00 |
| | 6.00 |
| 6 | |
| 10 | |

4. **To make 4.5 cups of fruity granola, the recipe calls for 1.5 cups of raisins, 1 cup of granola, and 2 cups of blueberries. If you want to make 18 cups of fruity granola, how much of each of the ingredients do you need?**

*Do you see how to set up proportions by using two different rows of the table?*



## Answers

1. A person who weighs 95 kilograms on Earth would weigh approximately 15.9 kilograms on the moon.
2. Water goes over Niagara Falls at a rate of 6 million cubic feet of water per minute.
3.

| Euro | U.S. Dollar |
|---|---|
| 1 | 1.07 |
| 0.93 | 1.00 |
| 5.61 | 6.00 |
| 6 | 6.42 |
| 10 | 10.7 |

4. 6 cups of raisins, 4 cups of granola, 8 cups of blueberries

## Answers

1. $n = \frac{3}{4}$
   Problem solutions will vary.
2. $h = 2$
   Problem solutions will vary.

NOTES

### TALK the TALK

## Choose Your Own Proportion Adventure

Write a problem situation for each proportion. Show the solution.

1. $\frac{8}{3} = \frac{2}{n}$

2. $\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{h}{1}$