Every Which Way

6

Combining Rigid Motions

MATERIALS

(optional)
Patty paper
Protractors
Straightedges

Lesson Overview

Students use coordinates to determine the rigid motion used to map one congruent figure onto another. They learn about and write congruence statements for congruent triangles. Using figures on a grid, students investigate and determine a sequence of transformations that can be used to verify figures are congruent. They then generalize the effects of rigid motions on the coordinates of figures.

Grade 8
Two-Dimensional Shapes

(10) The student applies mathematical process standards to develop transformational geometry concepts. The student is expected to:

(A) generalize the properties of orientation and congruence of rotations, reflections, translations, and dilations of two-dimensional shapes on a coordinate plane.
(C) explain the effect of translations, reflections over the *x*- or *y*-axis, and rotations limited to 90°, 180°, 270°, and 360° as applied to two-dimensional shapes on a coordinate plane using an algebraic representation.

ELPS

1.A, 1.C, 1.D, 1.E, 1.G, 2.C, 2.D, 2.G, 2.H, 2.I, 3.A, 3.B, 3.C, 3.D, 3.F, 4.A, 4.B, 4.C, 4.D, 4.G, 4.K, 5.E

Essential Ideas

- Reflections change the orientation of the vertices of a figure.
- Rigid motions produce congruent figures.
- Congruent line segments are line segments that have the same length.
- Congruent angles are angles that have equal measures.
- Congruent figures can be mapped from one to another through a sequence of translations, reflections, and rotations.
- There is often more than one sequence of transformations that can be used to map from one congruent figure onto another one.
- The effects of rigid motion transformations on the coordinates of figures can be generalized.

Lesson Structure and Pacing: 2 Days

Day 1

Engage

Getting Started: Going Backwards

Students use pre-images and images on the coordinate planes to determine the transformation applied to create each image. They reason using their knowledge of the properties of rigid motions and the effects of rigid motion transformations. Students also note that reflections change the orientation, or the order, of the vertices.

Develop

Activity 6.1: Congruence Statements

Students learn to write congruence statements for figures, segments, and angles. Congruent line segments and congruent angles are defined. A distinction is made between the symbol used to represent the actual measure of a line segment or an angle, and the symbol used to represent the geometric model of the object. Students use the idea of transformation mapping to correctly write congruence statements.

Activity 6.2: Using Rigid Motions to Verify Congruence

Students determine a sequences of rigid motion transformations that can map two congruent figures onto each other. First, students are given two images and pre-images of triangles. They identify the single transformation used, write a triangle congruence statement, and use the statement to list the congruent sides and congruent angles. Students then analyze three additional sets of figures to determine a sequence of transformations that verify the figures are congruent.

Day 2

Activity 6.3: Transformations with Coordinates

Students are given the coordinates a pre-image and the coordinates of its image. From this information they describe the single rigid motion or two rigid motion transformations used to form the image.

Demonstrate

Talk the Talk: Transformation Match-Up

Students create a table to summarize the effect of translations, reflections, and rotations on figures using coordinates.

Getting Started: Going Backwards

ENGAGE

Facilitation Notes

In this activity, students are given pre-images and images on the coordinate plane and identify the transformation applied to create each image.

Encourage students to reason about the transformations using their knowledge of the effects of rigid motions on the coordinates of the figures and on the properties of the image.

Have a student work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Questions to ask

- Was the figure reflected across the *x*-axis?
- Was the figure reflected across the *y*-axis?
- Was the figure horizontally translated?
- Was the figure vertically translated?
- Was the figure rotated 90° clockwise about the origin?
- Was the figure rotated 90° counterclockwise about the origin?
- Was the figure rotated 180° about the origin?

Summary

Rigid motion transformations performed on figures can be identified by mapping the corresponding coordinates of the pre-image onto the coordinates of the image. Reflections change the orientation, or the order of the vertices.

Activity 6.1
Congruence Statements

DEVELOP

Facilitation Notes

In this activity, students learn to write congruence statements for figures, segments, and angles. *Congruent line segments* and *congruent angles* are defined. A distinction is made between the symbol used to represent the actual measure of a line segment or an angle and the symbol used to represent the geometric model of the object. Students use the idea of transformation mapping to correctly write congruence statements.

Ask different students to read the introduction and definitions aloud. Review the Worked Examples and complete Questions 1 through 4 as a class.

Questions to ask

- What is the difference between the length of a line segment and a line segment?
- Are line segments considered congruent or equal?
- Are the lengths of line segments considered congruent or equal?
- Are the measures of line segments considered congruent or equal?
- What is the difference between the measure of an angle and an angle?
- Are angles considered congruent or equal?
- Are measures of angles considered congruent or equal?

Differentiation strategies

- Have students interact with the Worked Examples. Use two rods of equal length to demonstrate the differences among the statements. Have students measure the rods and explain that when referring to the numbers, an equal sign is used. Have students refer to the rods themselves and explain that when referring to shapes, a congruence symbol is used. Repeat the process using a cardboard angle for the second Worked Example.
- Summarize symbols to which students have been exposed ($=$, $\neq$, $\sim$, $\cong$) and have students compare and contrast when to use them.
- Emphasize the fact that sometimes it is easier to refer to a congruence statement rather than the congruent figures to determine what parts correspond or map onto each other. Have students take apart a congruence statement to name the 6 corresponding parts.

  For example: $\Delta NRJ \cong \Delta BLX$. By position in the congruence statement,

$$\Delta \text{NRJ} \cong \Delta \text{BLX}, \text{ so } \angle N \cong \angle B.$$
$$\Delta \text{NRJ} \cong \Delta \text{BLX}, \text{ so } \angle R \cong \angle L.$$
$$\Delta \text{NRJ} \cong \Delta \text{BLX}, \text{ so } \angle J \cong \angle X.$$
$$\Delta \text{NRJ} \cong \Delta \text{BLX}, \text{ so } \overline{NR} \cong \overline{BL}.$$
$$\Delta \text{NRJ} \cong \Delta \text{BLX}, \text{ so } \overline{RJ} \cong \overline{LX}.$$
$$\Delta \text{NRJ} \cong \Delta \text{BLX}, \text{ so } \overline{NJ} \cong \overline{BX}.$$

Emphasize that the order in which students name their segments matters, just as the orientation of the vertices matters after transformations. Students need to continue thinking of congruent figures as mappings of a pre-image onto an image, keeping in mind which points on the pre-image map to which points of the image.

Summary

Transformation mapping can be used to correctly write congruence statements.

Activity 6.2

Using Rigid Motions to Verify Congruence

Facilitation Notes

In this activity, students identify a sequences of rigid motion transformations that can map two congruent figures onto each other. They write a triangle congruence statement, and use the statement to list the congruent sides and congruent angles.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Differentiation strategy

Students may benefit from using patty paper as they engage in these questions.

As students work, look for

- Naming of triangles that do not reflect an accurate correspondence of parts.
- Attempts to determine corresponding sides by informally counting the slope of line segments from one vertex to another of a triangle.

Questions to ask

- How could you tell the transformation was not a translation?
- How could you tell the transformation was not a reflection?
- Do all of the transformations you have studied preserve both size and shape? Explain.
- What is another way to write the triangle congruence statement?

- If the triangle congruence statement is written differently, will that change the congruent parts of the triangle?
- How could you tell the transformation was not a rotation?

Have students work with a partner or in a group to complete Questions 3 through 5. Share responses as a class.

Note that these questions are more open than the previous two questions. Here, students can select which figure is the pre-image and which is the image. With congruent figures, it matters only that one can be mapped onto the other, not which image might be considered the pre-image.

Note that with regard to Question 5, it does not matter if $\overline{AB}$ maps to $\overline{JN}$, $\overline{NJ}$, $\overline{SP}$, or $\overline{PS}$. This allows for greater exploration and more sequences to be developed.

Questions to ask

- Which figure is the image?
- Which figure is the pre-image?
- Does it matter which figure is the image and which figure is the pre-image?
- Why doesn't it matter which figure is the image and which figure is the pre-image?
- Is there more than one correct way to write the congruence statement?
- What is another way to write the congruence statement?
- Was more than one rigid motion transformation used to create the image?
- Was the image a result of a translation? A rotation? A reflection?
- Is there another sequence of rigid motion transformations that could be used to create the same image?
- What is another sequence of rigid motion transformations that could be used to create the same image?

Summary

Sequences of transformations are used to verify the congruence of geometric figures.

Activity 6.3
Transformations with Coordinates

Facilitation Notes

In this activity, students are given the coordinates of a pre-image and the coordinates of its image. They describe the single rigid motion or combination of rigid motion transformations used to form the image.

Have students work with a partner or in a group to complete Questions 1 through 5. Share responses as a class.

Differentiation strategies

To scaffold support,

- Suggest they use graphs and patty paper and/or tables to visualize these problems.
- Generate a list of hints with the class, for starters:
  - If the orientation stays the same, a translation will work.
  - If a rotation is required, do that first, so it is easier to visualize the remaining required transformations.
  - Order does not matter in a sequence of transformations, either transformation can occur first.

Questions to ask

- What strategies did you use to determine the transformations for each question?
- Why are the images congruent to the pre-images?
- Which figure is the pre-image? The image?
- Is there more than one sequence of transformations in this situation?
- Was this figure reflected across the *x*-axis or the *y*-axis? How do you know?
- Was a translation involved in this situation? How do you know?
- Was a rotation involved in this situation? How do you know?
- What is the *x*-coordinate of each point in the pre-image? What is the *x*-coordinate of each point in the image?
- Is the relationship between the *x*-coordinate of each point in the pre-image and its corresponding *x*-coordinate in the image the same for all pairs of corresponding points?
- What is the *y*-coordinate of each point in the pre-image? What is the *y*-coordinate of each point in the image?
- Is the relationship between the *y*-coordinate of each point in the pre-image and its corresponding *y*-coordinate in the image the same for all pairs of corresponding points?

Summary

Images that result from rigid motion transformations are always congruent to the pre-image.

DEMONSTRATE

Talk the Talk: Transformation Match-Up

Facilitation Notes

In this activity, students use coordinates to describe the effects of translations, reflections, and rotations on figures.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Questions to ask:

- Which sets of coordinates describe translations?
- Which sets of coordinates describe reflections?
- Which sets of coordinates describe rotations?

Summary

When an arbitrary point (x, y) undergoes a horizontal or vertical translation of c units, the resulting coordinates are $(x + c, y)$, $(x - c, y)$, $(x, y + c)$, or $(x, y - c)$. When it undergoes a reflection across an axis, the resulting coordinates are $(x, -y)$ or $(-x, y)$. When it undergoes a 90° or 180° rotation about the origin, the resulting coordinates are $(y, -x)$, $(-y, x)$, or $(-x, -y)$.

Every Which Way

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Combining Rigid Motions

WARM UP

Determine the distance between each pair of points.

1. (2, 3) and (−5, 3)
2. (−1, −4) and (−1, 8)
3. (6, −2.5) and (6, 5)
4. (−8.2, 5.6) and (−4.3, 5.6)

LEARNING GOALS

- Use coordinates to identify rigid motion transformations.
- Write congruence statements.
- Determine a sequence of rigid motions that maps a figure onto a congruent figure.
- Generalize the effects of rigid motion transformations on the coordinates of two-dimensional figures.

KEY TERMS

- congruent line segments
- congruent angles

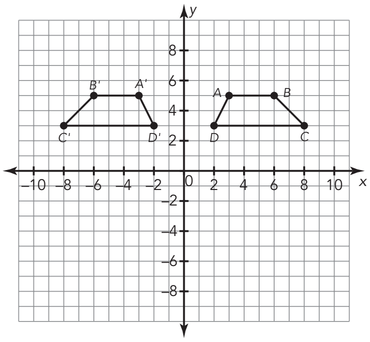
You have determined coordinates of images by translating, reflecting, and rotating pre-images. How can you use the coordinates of an image to determine the rigid motion transformations applied to the pre-image?

Warm Up Answers

1. The points are 7 units apart.
2. The points are 12 units apart.
3. The points are 7.5 units apart.
4. The points are 3.9 units apart.

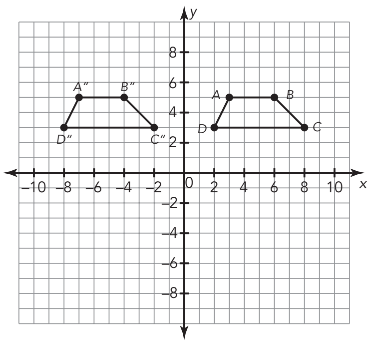
Answers

1a.



The figure was reflected across the *y*-axis. Explanations will vary.

1b.



The figure was translated 10 units to the left. Explanations will vary.

Getting Started

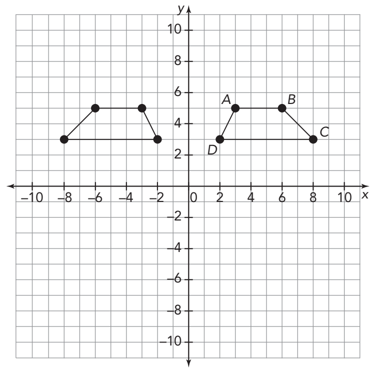
Going Backwards

Use your knowledge of rigid motions and their effects on the coordinates of two-dimensional figures to answer each question.

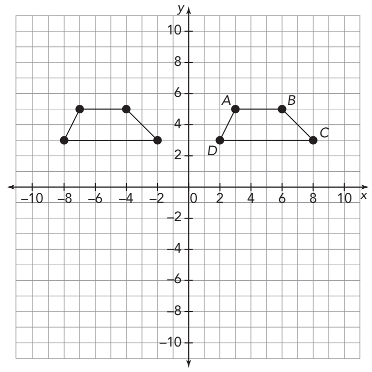
The line of reflection will be an axis, and the center of rotation will be the origin.

1. The pre-image and image of three different single transformations are given. Determine the transformation that maps the pre-image, the labeled figure, to the image. Label the vertices of the image. Explain your reasoning.

a.



b.



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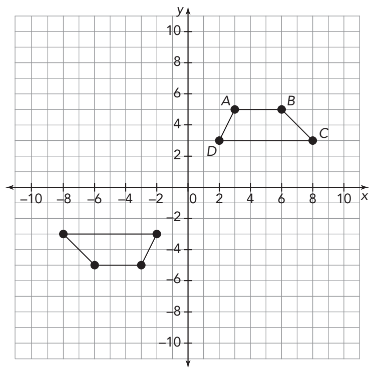
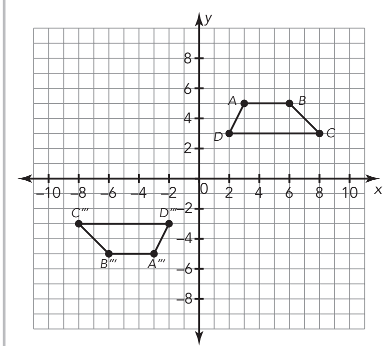
c. 

2. **Compare the order of the vertices, starting from *A*′, in each image with the order of the vertices, starting from *A*, in the pre-image.**

Answers

1c.



The figure was rotated 180° around the origin. Explanations will vary.

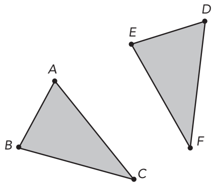
2. If I name the figures in clockwise order, all of the figures have vertices that proceed in alphabetical order except the reflection. In the reflection, the order of the vertices reversed.

Answers

1. Triangle *DEF* is a rotation of Triangle *ABC*.

ACTIVITY 6.1

Congruence Statements



You have determined that if a figure is translated, rotated, or reflected, the resulting image is the same size and the same shape as the original figure; therefore, the image and the pre-image are congruent figures.

1. How was Triangle *ABC* transformed to create Triangle *DEF*?

Congruent line segments are line segments that have the same length.

Because Triangle *DEF* was created using a rigid motion transformation of Triangle *ABC*, the triangles are congruent. Therefore, all corresponding sides and all corresponding angles have the same measure. In congruent figures, the corresponding sides are *congruent line segments*.

Think about congruent figures as a mapping of one figure onto the other. When naming congruent segments, write the vertices in a way that shows the mapping.

WORKED EXAMPLE

If the length of line segment *AB* is equal to the length of line segment *DE*, the relationship can be expressed using symbols. These are a few examples.

- $AB = DE$ is read "the distance between *A* and *B* is equal to the distance between *D* and *E*"
- $m\overline{AB} = m\overline{DE}$ is read "the measure of line segment *AB* is equal to the measure of line segment *DE*."

If the sides of two different triangles are equal in length, for example, the length of side *AB* in Triangle *ABC* is equal to the length of side *DE* in Triangle *DEF*, these sides are said to be congruent. This relationship can be expressed using symbols.

- $\overline{AB} \cong \overline{DE}$ is read "line segment *AB* is congruent to line segment *DE*."

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ELL Tip

Instruct students to read and take notes on *congruent line segments* and *congruent angles*. Provide students with a concept map to support this stage in their note-taking.

2. Write congruence statements for the other two sets of corresponding sides of the triangles.

Likewise, if corresponding angles have the same measure, they are *congruent angles*. **Congruent angles** are angles that are equal in measure.

WORKED EXAMPLE

If the measure of angle A is equal to the measure of angle D, the relationship can be expressed using symbols.

- $m\angle A = m\angle D$ is read "the measure of angle A is equal to the measure of angle D."

If the angles of two different triangles are equal in measure, for example, the measure of angle A in Triangle ABC is equal to the measure of angle D in Triangle DEF, these angles are said to be congruent. This relationship can be expressed using symbols.

- $\angle A \cong \angle D$ is read "angle A is congruent to angle D."

3. Write congruence statements for the other two sets of corresponding angles of the triangles.

You can write a single congruence statement about the triangles that shows the correspondence between the two figures. For the triangles in this activity, $\triangle ABC \cong \triangle DEF$.

4. Write two additional correct congruence statements for these triangles.

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Answers

2. $\overline{BC} \cong \overline{EF}$ and $\overline{AC} \cong \overline{DF}$
3. $\angle B \cong \angle E$ and $\angle C \cong \angle F$
4. Answers may vary. $\triangle BCA \cong \triangle EFD$ and $\triangle CAB \cong \triangle FDE$

Answers

1a. Triangle *TWC* was rotated 90° counterclockwise about the origin to create Triangle *PMK*.

1b. Answers may vary. $\triangle TWC \cong \triangle PMK$

1c. $\angle T \cong \angle P$, $\angle W \cong \angle M$, $\angle C \cong \angle K$

1d. $\overline{TW} \cong \overline{PM}$, $\overline{WC} \cong \overline{MK}$, $\overline{TC} \cong \overline{PK}$

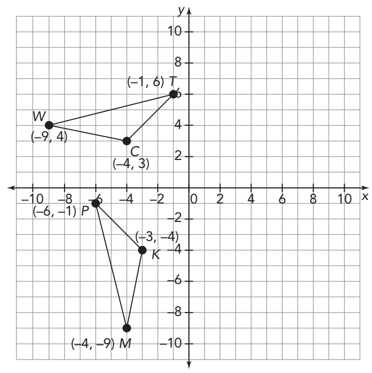
NOTES

ACTIVITY 6.2

Using Rigid Motions to Verify Congruence

You can determine if two figures are congruent by determining if one figure can be mapped onto the other through a sequence of rigid motions. Therefore, if you know that two figures are congruent, you should be able to determine a sequence of rigid motions that maps one figure onto the other.

1. Analyze the two congruent triangles shown.



a. Identify the transformation used to create ΔPMK from ΔTWC.

b. Write a triangle congruence statement.

c. Write congruence statements to identify the congruent angles.

d. Write congruence statements to identify the congruent sides.

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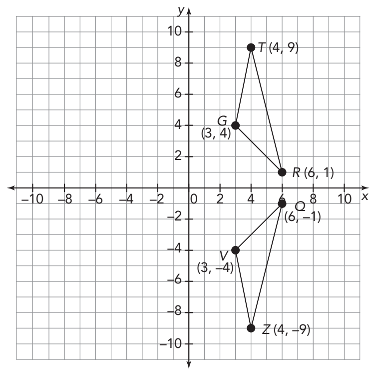
2. Analyze the two congruent triangles shown.



a. Identify the transformation used to create ΔZQV from ΔTRG.

b. Write a triangle congruence statement.

c. Write congruence statements to identify the congruent angles.

d. Write congruence statements to identify the congruent sides.

Answers

2a. $\triangle TRG$ was reflected across the x-axis to create $\triangle ZQV$.

2b. Answers may vary. $\triangle TRG \cong \triangle ZQV$

2c. $\angle T \cong \angle Z$, $\angle R \cong \angle Q$, $\angle G \cong \angle V$

2d. $\overline{TR} \cong \overline{ZQ}$, $\overline{RG} \cong \overline{QV}$, $\overline{TG} \cong \overline{ZV}$

Answers

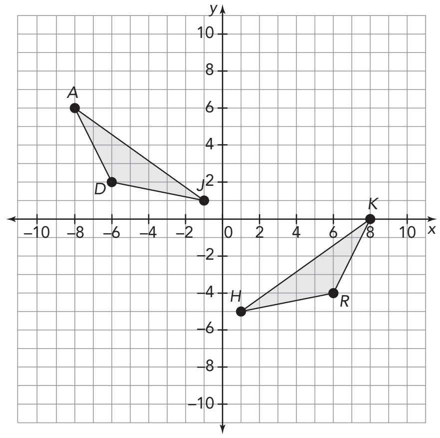
3a. $\triangle JAD \cong \triangle HKR$.
Explanations may vary.

3b. Answers will vary.
I can reflect Triangle *JDA* across the *y*-axis and then translate it down 6 units to map onto Triangle *HRK*.

3c. Yes, reversing these transformations is also a correct mapping.

3d. Because the orientations of the corresponding vertices are not the same, a reflection must be used to map from one triangle onto the other.

3. Analyze the two congruent triangles.



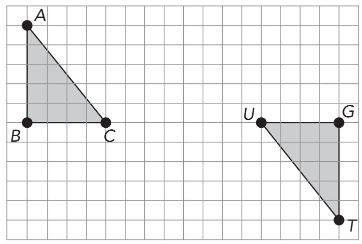
a. Write a congruence statement for the triangles. How did you determine the corresponding angles?

b. Identify a sequence of translations, reflections, and/or rotations that could be used to map one triangle onto the other triangle.

c. Reverse the order of the transformations that you used in part (b). Does this order map one figure onto the other?

d. Explain why it is not possible to map one figure onto the other using only rotations and translations.

4. Analyze the two congruent triangles.



a. Write a congruence statement for the triangles.

b. Identify a sequence of translations, reflections, and/or rotations that could be used to map one triangle onto the other triangle.

c. Reverse the order of the transformations that you used in part (b). Does this order map one figure onto the other?

d. Can you determine a way to map one triangle onto the other in a single transformation? Explain your reasoning.

NOTES

Answers

4a. $\triangle ABC \cong \triangle TGU$

4b. Answers may vary. Reflect Triangle *ABC* across a vertical line halfway between the triangles and then reflect Triangle $A'B'C'$ across the segment $\overline{B'C'}$.

4c. Yes, reversing these transformations is also a correct mapping.

4d. Answers may vary. Yes, you can determine a center of rotation (midpoint of the segment joining *C* and *U*) and rotate one figure 180° onto another.

Answers

5a. Answers will vary. One possibility is a rotation followed by a translation.

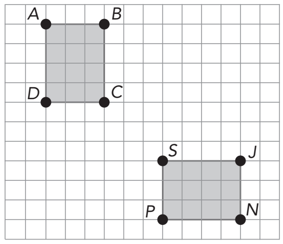
5b. Yes. You can locate a center of rotation and rotate Rectangle *ABCD* 90° counterclockwise. The center of rotation can be located by going to the midpoint of $\overline{BC}$ and to the right 5 spaces, or the midpoint of $\overline{SJ}$ and up 5 spaces.

Answers

1a. Triangle *JME* was formed by translating Triangle *PQR* 4 units to the left. Each *x*-value of the translated triangle is 4 less than the corresponding *x*-value of the original triangle.

1b. Triangle *DLG* was formed by translating Triangle *PQR* up 2.5 units. Each *y*-value of the translated triangle is 2.5 more than the corresponding *y*-value of the original triangle.

5. Analyze the two congruent rectangles.



a. Identify a sequence of translations, reflections, and/or rotations that could be used to verify that the rectangles are congruent.

b. Can you determine a way to map one rectangle onto the other in a single transformation? Explain your reasoning.

ACTIVITY 6.3

Transformations with Coordinates

For the triangles in this activity, $\Delta PQR \cong \Delta JME \cong \Delta DLG$.

1. Suppose the vertices of ΔPQR are *P* (4, 3), *Q* (−2, 2), and *R* (0, 0). Describe the translation used to form each triangle. Explain your reasoning.

a. *J* (0, 3), *M* (−6, 2), and *E* (−4, 0)

b. *D* (4, 5.5), *L* (−2, 4.5), and *G* (0, 2.5)

2. Suppose the vertices of ΔPQR are P (1, 3), Q (6, 5), and R (8, 1). Describe the rotation used to form each triangle. Explain your reasoning.

a. J (−3, 1), M (−5, 6), and E (−1, 8)

b. D (−1, −3), L (−6, −5), and G (−8, −1)

3. Suppose the vertices of ΔPQR are P (12, 4), Q (14, 1), and R (20, 9). Describe the reflection used to form each triangle. Explain your reasoning.

a. J (−12, 4), M (−14, 1), and E (−20, 9)

b. D (12, −4), L (14, −1), and G (20, −9)

4. Suppose the vertices of ΔPQR are P (3, 2), Q (7, 3), and R (1, 7).

a. Describe a sequence of a translation and reflection to form ΔJME with coordinates J (8, −2), M (12, −3), and E (6, −7).

b. Describe a sequence of a translation and a rotation to form ΔDLG with coordinates D (2, −6), L (3, −10), and G (7, −4).

5. Are the images that result from a translation, rotation, or reflection always, sometimes, or never congruent to the original figure?

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Answers

2a. Triangle *JME* was formed by rotating Triangle *PQR* 90° counterclockwise about the origin. Each ordered pair of Triangle *JME* is $(-y, x)$ in comparison to the ordered pairs of Triangle *PQR*.

2b. Triangle *DLG* was formed by rotating Triangle *PQR* 180° about the origin. Each ordered pair of Triangle *DLG* is $(-x, -y)$ in comparison to the ordered pairs of Triangle *PQR*.

3a. Triangle *JME* was formed by reflecting Triangle *PQR* across the y-axis. Each ordered pair of Triangle *JME* is $(-x, y)$ in comparison to the ordered pairs of Triangle *PQR*.

3b. Triangle *DLG* was formed by reflecting Triangle *PQR* across the x-axis. Each ordered pair of Triangle *DLG* is $(x, -y)$ in comparison to the ordered pairs of Triangle *PQR*.

4a. Triangle *JME* was formed by reflecting Triangle *PQR* across the x-axis and then translating to the right 5 units. Each ordered pair of Triangle *JME* is $(x + 5, -y)$ in comparison to the ordered pairs of Triangle *PQR*.

4b. Triangle *DLG* was formed by rotating Triangle *PQR* around the origin 90° clockwise and then translating down 3 units. Each ordered pair of Triangle *DLG* is $(y, -x - 3)$ in comparison to the order pairs of Triangle *PQR*.

5. All images that result from a translation, rotation, or reflection are always congruent to the original figure.

Answers

1. See table below.
2. Sample answer.
   A rotation of 180° is also a reflection across both axes. If I reflect across the x-axis first, I get $(x - y)$. Reflecting that point across the y-axis results in $(-x, -y)$. $(x, y) \rightarrow (x, -y) \rightarrow (-x, -y)$.

NOTES

TALK the TALK

Transformation Match-Up

Suppose a point (x, y) undergoes a rigid motion transformation. The possible new coordinates of the point are shown. Assume c is a positive rational number.

$(y, -x)$	$(x, y - c)$	$(x, -y)$
$(x + c, y)$	$(x - c, y)$	$(-y, x)$
$(-x, -y)$	$(-x, y)$	$(x, y + c)$

1. Record each set of new coordinates in the appropriate section of the table, and then write a verbal description of the transformation. Be as specific as possible.

Translations		Reflections		Rotations	
Coordinates	Description	Coordinates	Description	Coordinates	Description

2. Describe a single transformation that could be created from a sequence of at least two transformations. Use the coordinates to justify your answer.

1.

Translations		Reflections		Rotations	
Coordinates	Description	Coordinates	Description	Coordinates	Description
$(x + c, y)$	Translate to the right c units	$(x, -y)$	Reflect over the x-axis	$(y, -x)$	Rotate about the origin 90° clockwise
$(x - c, y)$	Translate to the left c units	$(-x, y)$	Reflect over the y-axis	$(-y, x)$	Rotate about the origin 90° counterclockwise
$(x, y + c)$	Translate up c units			$(-x, -y)$	Rotate about the origin 180°.
$(x, y - c)$	Translate down c units				