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The Squariest Square

From Informal to Formal Geometric Thinking

MATERIALS

Compasses
Patty paper
Protractors
Rulers
Straightedges

Lesson Overview

Through a series of activities, students consider the range of geometric reasoning from informal to formal. To start, students attempt to sketch a "perfect" square and discuss the properties of a square. They analyze a diagram with three squares, create specific angles within the squares, use a protractor to determine their measures, and compare the sum of the measures with their classmates' results. They consider a conjecture about the sum of the measures. To determine whether this conjecture holds true in a second case, they measure the angles in a larger version of the diagram. To move towards generalization, students use patty paper to further analyze the conjecture that the angle measures sum to 90°. The diagram is then expanded through rigid motions to create other geometric properties that students can consider to formally verify the proof, although this final step is not required. They conclude that informal reasoning involves measurements, while formal reasoning involves properties.

Geometry

Logical Argument and Constructions

(2) The student uses the process skills with deductive reasoning to understand geometric relationships. The student is expected to:

(A) distinguish between undefined terms, definitions, postulates, conjectures, and theorems.

(5) The student uses constructions to validate conjectures about geometric figures. The student is expected to:

(A) investigate patterns to make conjectures about geometric relationships, including angles formed by parallel lines cut by a transversal, criteria required for triangle congruence, special segments of triangles, diagonals of quadrilaterals, interior and exterior angles of polygons, and special segments and angles of circles choosing from a variety of tools.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I 3.D, 3.E, 4.B, 4.C, 4.D, 4.J, 5.B, 5.F, 5.G

Essential Ideas

- Mathematicians make conjectures, test predictions, experiment with patterns, and consider arguments and different perspectives.
- Mathematical reasoning can be used to validate a conjecture.

Lesson Structure and Pacing: 2 Days

Day 1

Engage

Getting Started: The Perfect Square

Students attempt to sketch a perfect square without using any tools. They compare their square with their classmates' squares, develop criteria to determine the most perfect square, and list properties of squares.

Develop

Activity 1.1: Analyzing a Diagram

Students are provided the distinction between the terms *sketch* and *draw*. Given a diagram composed of three squares, they draw additional line segments within the diagram to create specific angles. Students use a protractor to measure the angles. They compare the angle measures to those of their classmates and recognize that the angle measures appear to sum to 90°.

Day 2

Activity 1.2: Making Conjectures

Students are introduced to the term *conjecture*. They analyze a conjecture about the sum of the angle measures. Students investigate whether the conjecture is true in more cases by measuring the angles in a larger version of the diagram. They then consider how to determine the sum of the angle measures without using measuring tools. They then use patty paper to support their claim.

Activity 1.3: Drawing Auxiliary Lines

The diagram from the previous activity is shown with a translation of the three squares, creating a 2 by 3 grid, and auxiliary lines. Students analyze this enhanced diagram and generate a list of additional mathematical relationships they notice. The final step of formally proving the conjecture is not required for students.

Demonstrate

Talk the Talk: Proving Yourself

Students write a paragraph including examples to describe the differences that exist between informal geometric reasoning they have used in the past and formal thinking used to prove conjectures.

Getting Started: The Perfect Square

ENGAGE

Facilitation Notes

In this activity, students attempt to sketch a perfect square without using any tools. They compare their square with their classmates' squares, develop criteria to determine the most perfect square, and list properties of squares.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

As students work, look for

- Vocabulary such as *parallel, perpendicular, right angles, congruent,* and *equal*.
- Recognition of relationships among side measures, angle measures, and between sides (parallel and perpendicular).

Questions to ask

- What features do all squares have in common?
- What do you know about the measures of all four sides of a square?
- What do you know about the measures of all four angles of a square? What is another way to state that fact?
- What is the relationship between opposite sides of a square?
- What is the relationship between two consecutive sides of a square?
- What tools could you use to determine whether your sketch has the properties of a square?

Differentiation strategy

To extend the activity, expand the discussion to include other categories of polygons to which a square belongs, how to calculate the perimeter and area of a square, and any properties related to the diagonals of a square.

Summary

Geometric figures have defining properties.

DEVELOP

Activity 1.1
Analyzing a Diagram

Facilitation Notes

In this activity, students are provided the distinction between the terms *sketch* and *draw*. Given a diagram composed of three squares, they draw additional line segments within the diagram to create specific angles. Students use a protractor to measure the angles. They compare the angle measures to those of their classmates and recognize that the angle measures appear to sum to 90°.

Ask a student to read the introduction aloud. Discuss as a class.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

Differentiation strategy

To scaffold support, review segment and angle notation, proper naming of angles, and use of a protractor.

Questions to ask

- Describe the diagram that is provided.
- How would you identify each angle using 3 letters?
- What do the three angles have in common?
- Without measuring, can you tell what angle has the largest measure? Smallest measure? If so, explain.
- Why might you and your classmates have answers that are close but not exactly the same?
- Do you notice any relationship among the three angle measures? If so, explain.

Summary

Information regarding a specific case and recorded measurements provide insight regarding mathematical relationships. Measurements are not always exact.

Activity 1.2
Making Conjectures

Facilitation Notes

In this activity, students are introduced to the term *conjecture*. They analyze a conjecture about the sum of the angle measures. Students investigate whether the conjecture is true in more cases by measuring the angles in a larger version of the diagram. They then consider how to determine the sum of the angle measures without using measuring tools. They then use patty paper to support their claim.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

As students work, look for

Understanding of the terms *adjacent, congruent,* and *diagonals*.

Questions to ask

- Explain what a conjecture is in your own words.
- Why could you say that Jayda's statement is more general than the conclusion you made in the previous activity?
- Why does Jayda's thinking make sense?
- How did your results compare to those from the previous activity?

Have students work with a partner or in a group to complete Question 3. Share responses as a class.

As students work, look for

- The realization that the pieces of patty paper can be stacked and no additional tracing is required.
- How students verify that the sum of the three angles on the patty paper is 90°. Some may lay their angles in the corner of a square in the diagram. Others may use the corner of a page. How might they verify using the properties of the patty paper?

Questions to ask

- How did you manipulate your angles to show that their sum is 90°?
- Is there more than one solution? Explain.
- Does this strategy make it clear that the angle measures sum to exactly 90°? Why not?
- Can you complete this activity without knowing the measure of each individual angle? Explain.

Differentiation strategy

To scaffold support for all students, ask students how the patty paper activity would change if they had used the diagram in Activity 1.1. Have some students complete the activity using the smaller diagram and discuss why the results are the same.

Summary

Information regarding a specific case can sometimes be generalized to make a conjecture regarding a mathematical relationship.

Activity 1.3
Drawing Auxiliary Lines

Facilitation Notes

In this activity, the diagram from the previous activity is shown with a translation of the three squares, creating a 2 by 3 grid, and auxiliary lines. Students analyze this enhanced diagram and generate a list of additional mathematical relationships they notice. The final step of formally proving the conjecture is not required for students.

Ask a student to read the introduction aloud. Discuss as a class.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

Differentiation strategies

- To scaffold support,
  - Have them outline $\triangle AEG$. Then ask if they can recognize that triangle anywhere else in the diagram.
  - Have students use colored pencils or shading to visualize $\triangle AIK$, $\triangle AEG$, and $\triangle KLH$.
  - Suggest they mark congruent sides and angle measures in the diagram.
- To extend the activity, have students justify that $\triangle AKH$ is a right triangle. They can then show that $\angle KHA$ has the same measure as $\angle a$. Have students use this information to show that $\angle a + \angle b + \angle c = 90°$.

As students work, look for

Recognition that $\triangle AKH$ is a right triangle, even if they can't provide any justification.

Questions to ask

- What does it mean by *translate the three squares up*?
- What line segment in the original diagram is the same length as $\overline{AK}$ and $\overline{KH}$? How do you know?
- What other angles in the diagram can be labeled with b?
- What is an isosceles triangle? Do you recognize any isosceles triangles in the diagram?

Summary

An auxiliary line or line segment can be added to a diagram to help in solving or verifying a conjecture.

Talk the Talk: Proving Yourself

DEMONSTRATE

Facilitation Notes

In this activity, students write a paragraph including examples to describe the differences that exist between informal geometric reasoning they have used in the past and formal thinking used to prove conjectures.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

Questions to ask

- Are measurements enough to prove a conjecture? Explain why not.
- What is the value in taking measurements?
- What are examples of informal arguments?
- What are examples of formal reasoning you used in this lesson?

Summary

Informal reasoning can include measurements generated from a specific case. Formal reasoning requires the use of properties, definitions, and rules that prove that a conjecture is true in all cases.

NOTES

Warm Up Answers

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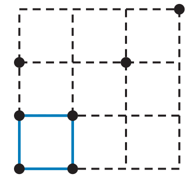

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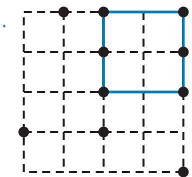

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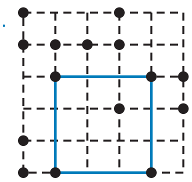

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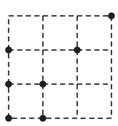
The Squariest Square

From Informal to Formal Geometric Thinking

Warm Up

Identify and connect the vertices that form a square in each grid.

1.


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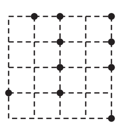

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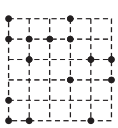


Learning Goals

- Recall properties of geometric figures.
- Understand that the results from measuring tools can be useful in composing a conjecture; however, they are not an acceptable form of mathematical reasoning to validate a conjecture.
- Make a geometric conjecture and use mathematical reasoning to validate it.

Key Terms

- sketch
- draw
- conjecture
- auxiliary line

You have reasoned about lines and shapes in earlier grades and courses. How can you apply formal geometric reasoning to what you know?

Answers

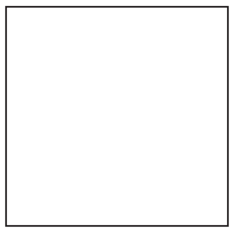
1. See students' sketches.
2. All the sides of the square should be the same length. All the angles of the square should be 90°.
3. Sample answers.
   All sides are congruent.
   All angles are 90°.
   Opposite sides are parallel.
   Consecutive sides are perpendicular.

GETTING STARTED

The Perfect Square

Can you sketch a perfect square freehand?

1. **Try to sketch a perfect square, like the one shown, without tracing or using tools.**



2. **Explain how you could decide whether one square is closer to "perfect" than another. Use your criteria to judge your and your classmates' best squares.**

3. **List some properties of squares that you know.**

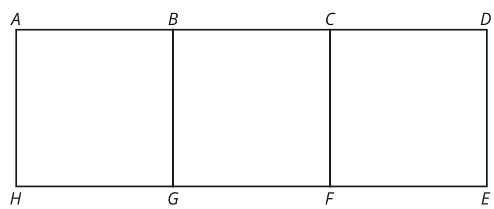
ACTIVITY **1.1**

Analyzing a Diagram

In a way, mathematical reasoning is not different from scientific reasoning. In mathematics, you come up with educated guesses and test them to see if they're correct. You can experiment with different patterns and consider arguments about mathematical statements. And, like other scientists, mathematicians gather evidence and become more and more confident about a statement when they obtain more evidence for it.

However, in mathematics, a statement is not true or false until it is proved to be true or false.

Consider the diagram composed of three adjacent squares.



1. Draw $\overline{AG}$, $\overline{AF}$, $\overline{AE}$. Then label $\angle AGH$ using the letter a, label $\angle AFG$ using the letter b, and label $\angle AEF$ using the letter c.

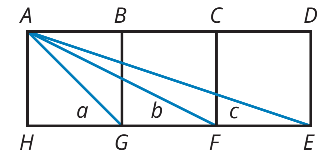
2. Use a protractor to measure $\angle a$, $\angle b$, and $\angle c$. List the angle measures.

3. Compare your measurements with your classmates' measurements. What do you notice?

When you **sketch** a geometric figure, you create the figure without tools. Accuracy is not important. When you **draw** geometric figures, you can use tools such as rulers, protractors, or a coordinate plane to draw exact lengths and areas.

Answers

1.



2. $m\angle a \approx 45°$

   $m\angle b \approx 26°$

   $m\angle c \approx 19°$

3. Sample answers.
   Our answers were close, but not exactly the same.
   The sum of the three angle measures seems to be 90°.

Answer

1. Sample answer.
No, I don't think the size of the squares will affect the angle measures. All angles in a square are 90°, and the diagonal line segments drawn to form $\angle a$, $\angle b$, and $\angle c$ were drawn the same way.

ACTIVITY **1.2**

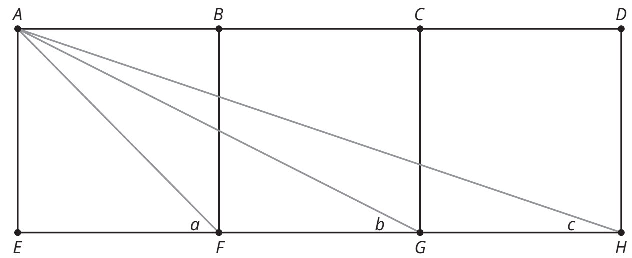
Making Conjectures

A **conjecture** is a mathematical statement that appears to be true, but has not been formally proved.

In the previous activity, you may have noticed that the sum of the measures of $\angle a$, $\angle b$, and $\angle c$ is close to or equal to 90°. Jayda made a *conjecture* about the sum of the angle measures.

> Jayda
> The size of the squares doesn't matter. Given any three adjacent and congruent squares, if the diagonals are drawn in the same way, the sum of the angle measures will always be 90°.

Let's consider a diagram of three differently-sized adjacent and congruent squares. The same lines are drawn and triangles formed.



1. Without measuring, do you think the size of the squares will affect the sum of the measures of $\angle a$, $\angle b$, and $\angle c$? Explain your thinking.

2. **Use a protractor to test your prediction on these differently-sized squares. Record your results.**

Ask yourself:

Do your results support Jayda's conjecture?

There are many different ways to verify that the sum of the angle measures could be 90°. Experimenting with different methods and visualizing are important tools that mathematicians use to approach problems in effective ways and gain confidence in their conclusions.

3. **Copy each of the angles *a*, *b*, and *c* from the diagram onto a different piece of patty paper. How can you manipulate the three angles to show that their sum is 90°?**

Answers

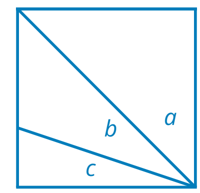
2. $m\angle a \approx 45°$

   $m\angle b \approx 26°$

   $m\angle c \approx 19°$

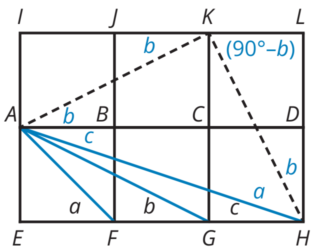
   My measurements were the same as the ones in the smaller diagram.

3. Sample answer.



Answer

1. Sample answers.



Triangles *AIK*, *AEG*, and *KLH* are all right triangles because each triangle has an angle that is also a corner of a square.

$\overline{AG}$ is the hypotenuse of $\triangle AEG$. $\overline{AK}$ is the hypotenuse of $\triangle AIK$. $\overline{HK}$ is the hypotenuse of $\triangle KLH$. Each triangle has a leg one unit in length and a leg two units in length. According to the Pythagorean Theorem, each hypotenuse would have the same length.

$\angle L$ is 90° and $\angle KHL$ is b°, so angle $\angle LKH$ must equal $(90 - b)$° for the angle measures to sum to 180°.

$\angle AKH$ is 90°. Angles *JKA*, *AKH*, and *LKH* form a straight line totaling 180°. The measures of $\angle JKA$ and $\angle LKH$ total 90°, so $\angle AKH$ must be a 90° angle.

$\triangle AEF$, $\triangle AKH$, and $\triangle AKG$ are isosceles triangles.

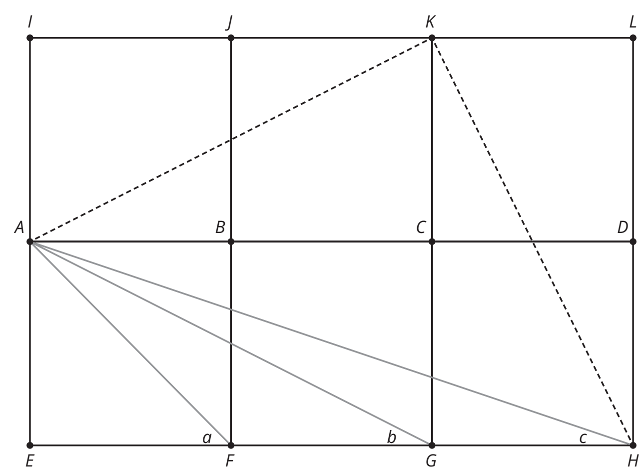
ACTIVITY 1.3

Drawing Auxiliary Lines

An **auxiliary line** is a line or line segment added to a diagram to help in solving or proving a concept.

Making arguments about statements in mathematics may seem like a rigid process at times, but it can also involve a lot of creativity. You can, for example, draw extra lines, called *auxiliary lines*, and perform rigid motions like translations, reflections, and rotations when you are reasoning geometrically.

Let's consider a new diagram that is the result of translating the original three squares up and drawing two auxiliary line segments.



1. What other angle measures or side lengths can you determine using these added figures? List all the concepts and facts you use.

NOTES

TALK the TALK

Proving Yourself

In this course, you will move from making conjectures and creating informal arguments to proving, for good, that certain mathematical statements must be true. You will learn to use properties and definitions to prove or disprove many conjectures.

1. **Write a paragraph describing the differences that you think exist between the informal geometric reasoning you have used in the past and the formal thinking used to prove conjectures. Use examples to illustrate your answer.**

Answer

1. Sample answers. Informal reasoning can include measurements generated from a specific case. Formal reasoning requires the use of properties, definitions, and rules that prove a conjecture is true in all cases.