2

Bow Thai

Translations as Functions

MATERIALS

Compasses
Patty paper
Straightedges

Lesson Overview

Students analyze transformation machines and conclude that translations along parallel lines always produce images that are congruent to their pre-image, while translations along rays with a common endpoint produce dilations or images that are similar, but not congruent, to their pre-image. The term *isometry* is defined to label these differences, with the understanding that any rigid motion transformation that preserves size and shape is an isometry. Students then engage in a context involving an animated website where they learn and use function notation to represent geometric translations.

Geometry

Coordinate and Transformational Geometry

(3) The student uses the process skills to generate and describe rigid transformations (translation, reflection, and rotation) and non-rigid transformations (dilations that preserve similarity and reductions and enlargements that do not preserve similarity). The student is expected to:

(B) determine the image or pre-image of a given two-dimensional figure under a composition of rigid transformations, a composition of non-rigid transformations, and a composition of both, including dilations where the center can be any point in the plane.

(C) identify the sequence of transformations that will carry a given pre-image onto an image on and off the coordinate plane.

Proof and Congruence

(6) The student uses the process skills with deductive reasoning to prove and apply theorems by using a variety of methods such as coordinate, transformational, and axiomatic and formats such as two-column, paragraph, and flow chart. The student is expected to:

(C) apply the definition of congruence, in terms of rigid transformations, to identify congruent figures and their corresponding sides and angles.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 4.D, 4.J, 5.B, 5.F, 5.G

Essential Ideas

- Translations along parallel lines are rigid motions and always produce images that are congruent to the pre-image.
- A translation is a function, represented as $T_{AB}(P) = P'$ which takes as its input the location of a point P and translates it a distance AB in the direction AB.
- Isometries are rigid motion transformations that preserve size and shape.

Lesson Structure and Pacing: 2 Days

Day 1

Engage

Getting Started: Universal Translator

Students analyze three transformation machines with different relationships among the lines and rays involved. They conclude that translations along parallel lines always produce images that are congruent to their pre-image, while translations along rays with a common endpoint produce dilations or images that are similar, but not congruent, to their pre-image.

Develop

Activity 2.1: Translation Functions

Students are presented with a context involving an image on a website that needs to be translated. They analyze Worked Examples that demonstrate how to use a directed line segment to complete a translation and how to use function notation to represent translations. They then complete translations represented using function notation. The terms *translation* and *isometry* are introduced.

Day 2

Activity 2.2: Determining Congruence Using Translations

Students continue to use the animated website context to write and draw translation functions in order to demonstrate that triangles are congruent. They then use a different example involving a dilation to distinguish between dilations and isometries.

Demonstrate

Talk the Talk: Isometries on the Menu

Students distinguish between an isometry and a transformation that is not an isometry. They write functions to represent given translations and compare geometric translation functions and algebraic equations that represent the translation of a function.

Getting Started: Universal Translator

Facilitation Notes

In this activity, students analyze three transformation machines with different relationships among the lines and rays involved. They conclude that translations along parallel lines always produce images that are congruent to their pre-image, while translations along rays with a common endpoint produce dilations or images that are similar, but not congruent, to their pre-image.

Have students work with a partner or in a group to complete Questions 1 through 4. Share responses as a class.

As students work, look for

- The use of patty paper or a compass and straightedge to translate the fixed distance.
- Use of mathematical vocabulary, such as *congruent*, *similar*, and *dilation*.
- Different methods used to determine that a similar figure was produced by the translation in Question 2.

Questions to ask

- How can you label the lines to identify that they are parallel?
- What is the relationship among the pre-image and image in Question 1?
- Would the image in Question 1 be different if parallel rays or segments were used? Why not?
- What is the relationship among the rays in Question 2?
- What is the relationship among the pre-image and image in Question 2?
- How do you know that the image is similar to the pre-image?
- Would the image in Question 2 be different if lines or segments were used? Why not?
- What is the difference between the outputs of the translations in Questions 1 and 2?
- Why do you think the images in Questions 1 and 2 are not identical?
- How is Question 3 different from Questions 1 and 2?
- What is the relationship between the pre-image and image in Question 3?
- What is the meaning of the term *dilation*?
- What effect does the relationship among the orientation of the lines, rays, or segments have on the output of a translation?

Differentiation strategies

- To assist all students, make it explicit that the purpose of this activity is to refine their informal understanding of translations and rigid motions. All rigid motions preserve size and shape. A translation along parallel lines is a rigid motion that preserves both size and shape. A translation along non-parallel lines is not a rigid motion.
- To extend the activity,
  - Have students complete another translation in Question 2 that is the distance *AB* to the left of the pre-image. Discuss the effects.
  - Instruct students to decompose the pre-image in Question 3 to top and bottom triangle pre-images. Discuss the differences in the images produced and why they occur.

Summary

Translations along parallel lines are rigid motions and always produce images that are congruent to the pre-image. Translations along rays with a common endpoint are not rigid motions and produce dilations or images that may be similar to, but not congruent to, the pre-image.

Activity 2.1

Translation Functions

DEVELOP

Facilitation Notes

In this activity, students are presented with a context involving an image on a website that needs to be translated. They analyze Worked Examples that demonstrate how to use a directed line segment to complete a translation and how to use function notation to represent translations. They then complete translations represented using function notation. The terms *translation* and *isometry* are introduced.

Ask a student to read the introduction aloud. Discuss as a class.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

As students work, look for

The different methods students use to measure the length of line segments.

Questions to ask

- How did you determine the locations of points *S*′, *Q*′, and *R*′?
- How did you measure the lengths of the line segments used in the translation?

- Should the line segments be the same length? Why or why not?
- How do you know $\overline{QQ'}$ is parallel to $\overline{SS'}$ and $\overline{RR'}$?

Differentiation strategy

To assist all students, have them select another point on $\triangle SQR$ and demonstrate how that point is translated the exact same distance along a parallel line to locate another point on $\triangle S'Q'R'$.

Have students work with a partner or in a group to analyze the Worked Example and complete Questions 2 and 3. Share responses as a class.

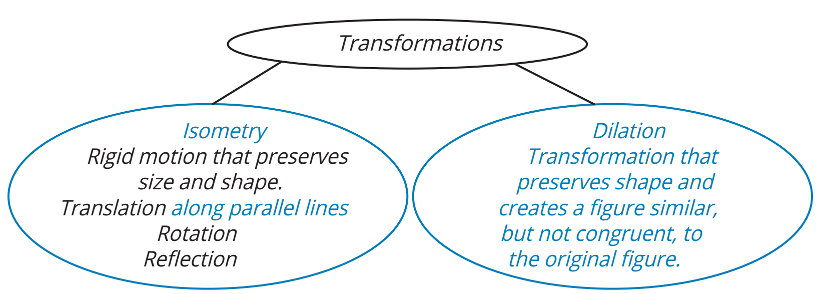
Questions to ask

- What is meant by a directed line segment?
- What are the vertical and horizontal distances of the translation as noted by the directed $\overline{AB}$?
- Do you think the horizontal and vertical distances or the total distance was used to locate each point on the corresponding figure? Why?
- How do you think the direction of $\overline{AB}$ was determined?
- Why is the distance between each pair of corresponding points equal?
- How did you determine the distance between points A and B?
- How can the Pythagorean Theorem be used to determine the distance between points A and B?
- Where did you locate another directed line segment on the grid? Could it be anywhere on the grid?
- Is there more than one way to write the triangle congruence statement? What is another way?
- Is the congruence statement, $\triangle MNP \cong \triangle P'M'N'$ a true statement? Why or why not?
- Why are there 6 equality statements and 6 congruency statements?
- What are the 6 equality statements and 6 congruency statements?

Ask a student to read the definitions following Question 3 aloud. Analyze the Worked Example and complete Questions 4 and 5 as a class.

Differentiation strategy

To extend the activity, revisit the framework provided in the previous lesson to summarize the difference between rigid motions and dilations.



Note: A dilation of 1 does not change the size of the figure.

Questions to ask

- How does a translation function represent the distance of the translation?
- How does a translation function represent the direction of the translation?
- Do the distance and direction specified by the subscript: T_{AB} mean to translate along line segment *AB* in the direction from *A* to *B* and the distance from *A* to *B*?
- Would the translation be any different if the subscripts were written as *BA*? If so, how?
- Why is P'' to the left of *P*?
- What does the notation $T_{AB}(P) = P'$ represent?
- What does the notation $T_{AC}(P) = P''$ represent?
- How did you identify the distance and direction of the translation of $\triangle SQR$? Does it matter what letters you use? Does the order of the letters matter?

Have students work with a partner or in a group to complete Questions 6 through 8. Share responses as a class.

As students work, look for

- Confusion with the notation T_{AB} (Sun). The entry in parentheses represents the figure to be translated. If the entry applies to a point or each individual point of a figure, a capital letter should be used. It is also acceptable to use the vertices of a figure or the name of a figure. Students will see in Lesson 4 that a lowercase letter can also be used to represent a figure. Stress the importance of the accuracy of the subscript, and allow various correct responses for the parentheses component.
- The use of a line parallel to $\overline{AB}$ when translating the point *P* on the sun the distance *AB* and the use of a line parallel to $\overline{L'M}$ when translating the point *P* on the moon the distance *L'M*.
- The use of construction tools for accuracy of parallel lines.
- The use of patty paper to trace the entire figure.

Differentiation strategy

To scaffold support with translating the figures in Question 7, demonstrate how to use patty paper to translate the sun, then have students duplicate the process to translate the moon.

- Connect points *A* and *B* to identify the directed line segment.
- Use a straightedge to extend the line segment beyond point *B*.
- Trace the entire diagram, including the sun and points *P, A,* and *B* on patty paper.
- Slide the entire diagram along the extended line until point *A* lies on point *B*.

- The sun is now translated to its new location according to the translation function.

Then, discuss why this method using patty paper is an effective method for completing a translation.

Questions to ask

- Why is Greta correct?
- What is being translated, the sun or a line segment? How do you know?
- What is the purpose of point *P*?
- Should the image of the sun be above or below the pre-image? How do you know?
- Using the notation $T_{AB}(Sun)$, point *P* is moved what distance? What direction?
- Should the image of the moon be to the right or to the left of the pre-image? How do you know?
- Using the notation $T_{L'M}(Moon)$, point *P* is moved what distance? What direction?
- Does it make any difference that the line segment intersects the diagram of the moon? Why or why not?
- What is the relationship between the slopes of parallel lines? How is this relationship helpful when performing translations?

Summary

A translation is a function, represented as $T_{AB}(P) = P'$, which takes as its input the location of a point *P* and translates it a distance *AB* in the direction *AB*. Because these translation functions move a set of points along parallel lines, they are isometries or rigid motions that preserve size and shape.

Activity 2.2

Determining Congruence Using Translations

Facilitation Notes

In this activity, students continue to use the animated website context to write and draw translation functions. They then use a different example involving a dilation to distinguish between dilations and isometries.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

Differentiation strategy

To assist all students, suggest that they label the vertices of each triangle to assist in writing the translation functions.

As students work, look for

- Different labeling of the vertices.
- The selection of different vertices to demonstrate the directed line segment.
- An understanding that the use of directed line segments provides translations along parallel lines.
- Vocabulary that demonstrates that difference between isometries and dilations.

Questions to ask

- How does your translation function represent the distance of the translation?
- How does your translation function represent the direction of the translation?
- What is another way to represent this translation function?
- How did you draw the directed line segment representing this translation function?
- Why don't all the directed line segments look exactly alike?
- What does a directed line segment have to do with parallel lines?
- What do parallel lines have to do with translations? With isometries?
- If the three triangles are no longer congruent to the triangle background behind the word menu, are the transformations that created them isometries?
- How does your process demonstrate that this transformation is not an isometry?
- How could you demonstrate that the background triangle is similar to the original triangle?

Summary

Isometries preserve shape and size; they create congruent figures. Dilations preserve shape only; they create similar figures.

Talk the Talk: Isometries on the Menu

DEMONSTRATE

Facilitation Notes

In this activity, students distinguish between an isometry and a transformation that is not an isometry. They write functions to represent translations and compare geometric translation functions and algebraic equations that represent the translation of a function.

Have students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

Questions to ask

- What is the definition of an isometry?
- What is an example of an isometry?
- How is rigid motion related to an isometry?
- Are all transformations also isometries?
- What is an example of a transformation that is not an isometry?
- Describe a translation that is an isometry and a translation that is not an isometry.
- Does the order of the subscripts make a difference?
- How did you represent the trapezoid or triangle in your function?
- Provide an example of a geometric translation function.
- Provide an example of an algebraic equation which shows the translation of a function.
- How would you compare the inputs? Notation? Movement?

Misconception

While students may recognize that geometric translation functions and algebraic equations that show the translation of a function both represent movements in a plane, they may not consider the variance in the movements that can be represented by the notation. Geometric translation functions are described as movements that follow a directional line segment that is not limited by horizontal and vertical movements. Algebraic equations that represent the translation of a function are limited to horizontal and/or vertical movements.

Differentiation strategy

To extend the lesson, ask students to translate a line segment using a geometric translation function and an algebraic equation which shows the translation of a function.

Summary

Isometries maintain congruence of the input figure and output figure, or pre-image and image.

2

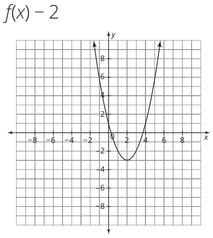
Bow Thai

Translations as Functions

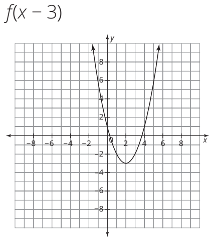
Warm Up

Each coordinate plane shows the graph of $f(x)$. Sketch the graph of $g(x)$.

1. $g(x) = f(x) - 2$



2. $g(x) = f(x - 3)$



Learning Goals

- Represent translations on the plane.
- Describe translations as functions that take points on the plane as inputs and produce translated points as outputs.
- Compare transformations that preserve distance and angles, called isometries, to transformations which are not isometries.

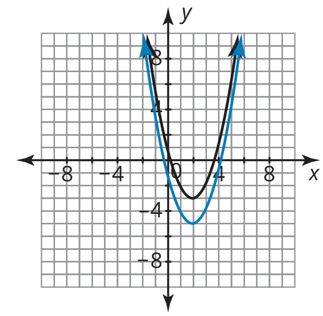
Key Terms

- translation
- isometry

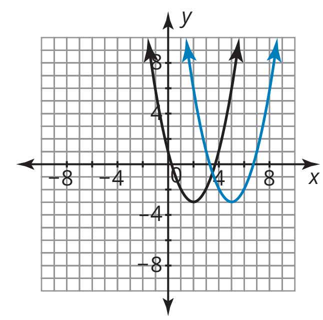
You have learned how to represent the horizontal or vertical translation of a function. How can you write geometric translations of figures on the plane as functions?

Warm Up Answers

1.

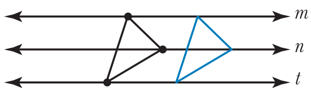


2.

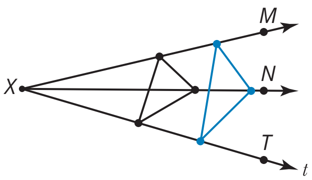


Answers

1.



2.

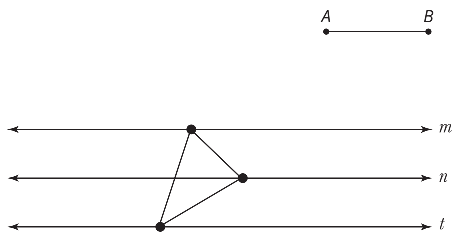


GETTING STARTED

Universal Translator

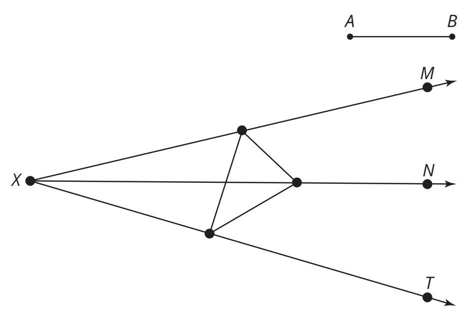
Recall that you used translations in the transformation machine in Lesson 1: *Put Your Input In, Take Your Output Out*. You translated figures along straight lines or line segments.

1. Lines *m*, *n*, and *t* are parallel lines. Draw a translation of each vertex of the triangle along the line the point is located on. Translate each point to the right along the line a distance equal to *AB*. Connect the points to form a triangle.

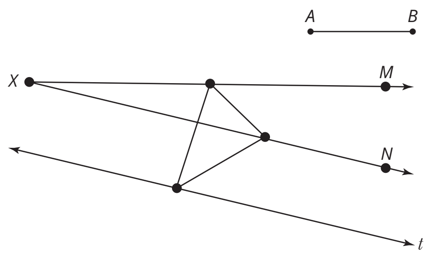


Points *M*, *N*, and *T* are necessary to name each ray.

2. Rays *XM*, *XN*, and *XT* share a vertex point. Draw a translation of each vertex of the triangle along the ray the point is located on. Translate each point to the right along the ray a distance equal to *AB*. Connect the points to form a triangle.



3. **Rays *XM* and *XN* share a vertex point, and line *t* is parallel to $\overrightarrow{XN}$. Draw a translation of each vertex of the triangle along the ray or line the point is located on. Translate each point to the right along the ray or line a distance equal to *AB*. Connect the points to form a triangle.**



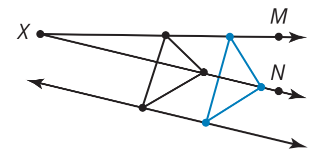
4. **Compare the diagrams you created in Questions 1 through 3. Each set of three lines and/or rays makes up a transformation machine.**

   a. **Which transformation machine produces a translation of the triangle? Explain your reasoning.**

   b. **Which transformation machine produces a dilation of the triangle? Explain your reasoning.**

Answers

3. Check students' drawings. Answers are not drawn to scale.



4a. The transformation machine in Question 1 produces a translation because the image of the triangle is congruent to the original and in the same orientation.

4b. The transformation machine in Question 2 produces a dilation because the image of the triangle is proportional to the original triangle.

Answers

1a. Check students' drawings.

1b. The line segments are all parallel to each other.

1c. The segments are the same length.

1d. QQ' is the same distance as SS' and RR'.

The line containing $\overline{QQ'}$ is parallel to the line containing $\overline{SS'}$ and $\overline{TT'}$.

ACTIVITY **2.1**

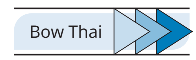
Translation Functions

Transformations are used frequently in web design and game animation and are often written as functions, which take points, distances, and angles as inputs. The functions output a new set of points after applying a transformation. These transformations move objects around on the screen.

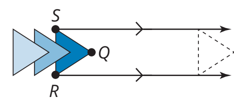
Think about:

A translation moves a set of points a specified distance in a specified direction along parallel lines.

Suppose you are designing a website banner for a new restaurant. The banner will show three congruent triangles animated from left to right, and then the name will fade in.



1. Consider the translation of the first triangle, $\triangle SQR$.



a. Label the points of the image, $\triangle S'Q'R'$.

b. What relationship is there between $\overline{SS'}$ and $\overline{RR'}$?

c. Measure the lengths of the two line segments used in the translation. What do you notice?

d. What do you know about the distance QQ'? What do you know about the line containing $\overline{QQ'}$?

ELL Tip

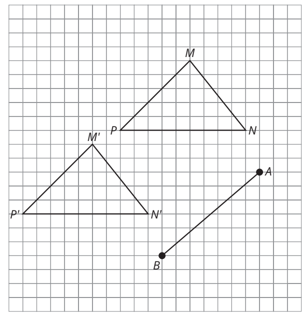
Two non-mathematical terms that appear in this activity are *web design* and *game animation*. Create an anchor chart displaying the definition of each term. Define *web design* as *the process of creating websites*. Define *game animation* as *the process of creating movement and behavior of characters and objects*. Discuss these terms in the context of the scenario at the beginning of the activity. Ensure students' understanding and clarify any remaining misunderstandings of the terms.

Worked Example

A translation can be measured as a directed line segment.

$\triangle MNP$ was translated to produce $\triangle M'N'P'$. The triangle was translated a distance equal to the distance between points A and B. It was translated in the direction from point A to point B.

So, $\overline{AB}$ is the directed line segment used to measure this translation.



2. Suppose each grid square is 1 unit × 1 unit.

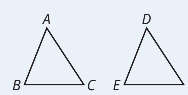
a. What is the distance from point *A* to point *B*?

b. Compare the distance *AB* with the distances *MM'*, *NN'*, and *PP'*. What do you notice?

c. Can you draw another directed line segment on the grid which defines the translation of $\triangle MNP$ to $\triangle M'N'P'$? If so, draw the segment on the grid. Explain your thinking.

3. Write equality and congruence statements to compare the corresponding sides and angles of the pre-image $\triangle MNP$ and the image $\triangle M'N'P'$.

Remember:



If $\triangle ABC \cong \triangle DEF$, then:

$\overline{AB} \cong \overline{DE}$

$AB = DE$

$\angle ABC \cong \angle DEF$

$m\angle ABC = m\angle DEF$

Answers

2a. The distance $AB = \sqrt{6^2 + 7^2} \approx 9.22$ units.

2b. All the distances are equal.

2c. Check the placement of students' line segments. The line segment should be parallel to $\overline{AB}$ and the same length as $\overline{AB}$.

3. $MN = M'N'$

$MP = M'P'$

$NP = N'P'$

$m\angle M = m\angle M'$

$m\angle N = m\angle N'$

$m\angle P = m\angle P'$

$\overline{MN} \cong \overline{M'N'}$

$\overline{MP} \cong \overline{M'P'}$

$\overline{NP} \cong \overline{N'P'}$

$\angle M \cong \angle M'$

$\angle N \cong \angle N'$

$\angle P \cong \angle P'$

Answers

4. The direction and distance are specified by the subscript of the function. T_{AB} means to translate in the direction from *A* to *B* and the distance *AB*.

5a. $T_{SS'}(\triangle SQR) = \triangle S'Q'R'$

$T_{QQ'}(\triangle SQR) = \triangle S'Q'R'$

$T_{RR'}(\triangle SQR) = \triangle S'Q'R'$

5b. Sample answer. Every point on $\triangle SQR$ moves a distance equal to SS' in the direction $\overline{SS'}$.

Remember:

A function is a rule that assigns exactly one output to each input.

Inputs to functions do not have to be numbers. They can be points, too.

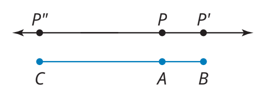
A **translation** is a function, *T*, which takes as its input a set of pre-image points and outputs a set of image points. The pre-image points are translated a distance of *AB* in the direction *AB*. For example, a translation of point *P* could be expressed as $T_{AB}(P)$, or *P*′. A translation is an example of an *isometry*. An **isometry** is a rigid motion transformation that preserves size and shape.

Worked Example

A translation function can represent the distance and direction of the translation using a line or line segment, or a parallel line or line segment.

$T_{AB}(P) = P'$

$T_{AC}(P) = P''$



4. Identify how the distance and direction of the translation are specified in each of the functions.

5. Consider the translation of the website banner from Question 1.

a. Use the notation from the Worked Example to describe the translation of △*SQR*.

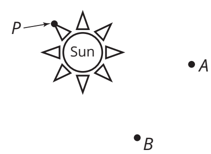
b. Explain how your function represents the translation of every point of △*SQR*.

6 • TOPIC 2: Rigid Motions on a Plane

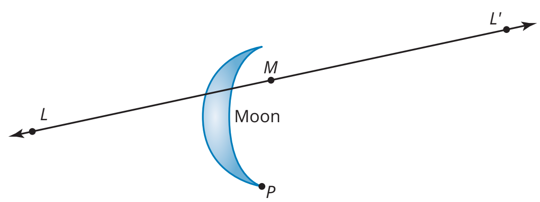
6. Greta says that the exact same function can be used for every triangle in the animated web banner. Is Greta correct? Explain your thinking, and then draw the translations to justify your answer.

7. Complete each translation using the given function.

a. T_{AB}(Sun)



b. $T_{L'M}$ (Moon)



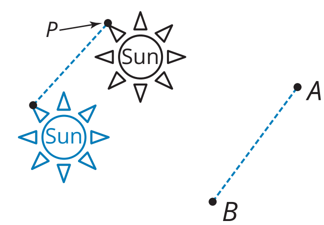
8. Explain why you can use parallel lines when describing translations.

Answers

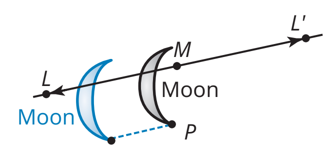
6. Greta is correct. The set of 3 triangles together is translated according to the translation function given in Question 5, part (a).

7. Check students' drawings. Answers are not drawn to scale.

7a.



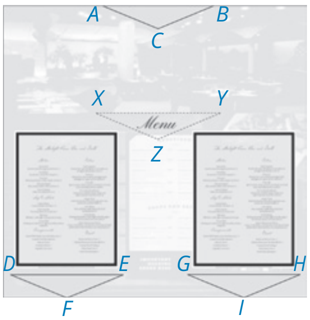
7b.



8. Parallel lines determine the horizontal and vertical translations.

Answers

1. Sample answers.



$T_{AX}(\triangle ABC) = \triangle A'B'C'$

$T_{DX}(\triangle DEF) = \triangle D'E'F'$

$T_{GX}(\triangle GHI) = \triangle G'H'I'$

Check students' drawings of the directed line segment for each translation.

2. Sample answers.
All three triangles in the original menu are congruent, and they are all translated along parallel lines, so their translations are rigid motions which preserve size and shape.
Each triangle is congruent to $\triangle XYZ$.

ACTIVITY **2.2**

Determining Congruence Using Translations

When users click on the menu of Bow Thai, copies of 3 triangles will move from the corners and top of the web page to the center to form the background behind the word "Menu" as shown.

1. Write and draw translation functions to show how each triangle will move on the page.

2. Are the triangles all congruent? Explain why or why not.

The owner of Bow Thai is thinking about using smaller triangles on the sides of the menu web page. She still wants the triangles to move and merge to form the triangle background behind the word "Menu."

3. Would these transformations be isometries? Demonstrate why or why not and explain your process.

Answer

3. The transformations are not isometries because the transformed figures have longer sides than the original figures.

Answers

1. An isometry is a transformation that preserves both size and shape. A translation along parallel lines is an example of an isometry. A transformation that is not an isometry does not preserve size and shape. A dilation and a translation along non-parallel lines are examples of transformations that are not isometries.

2a. Sample answer. T_{PQ} (Trapezoid $ABCD$) = Trapezoid $A'B'C'D'$

2b. Sample answer. T_{QP} ($\triangle XYZ$) = $\triangle X'Y'Z'$

3. Sample answers. Geometric translation functions and algebraic equations which show the translation of a function both represent movements in a plane. Geometric translation functions are described as movements that follow a directional line segment that is not limited by horizontal and vertical movements. Algebraic equations that represent the translation of a function are limited to horizontal and/or vertical movements. Geometric translations move shapes, while algebraic equations move functions on a coordinate plane. Geometric translations and algebraic equations use different notations.

NOTES

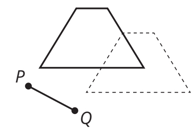
TALK the TALK

Isometries on the Menu

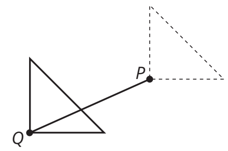
1. **Describe how to distinguish between an isometry and a transformation that is not an isometry.**

2. **Write a function to describe each translation. Each solid figure is a pre-image, and each dashed figure is an image.**

a.



b.



3. **What similarities and differences are there between a geometric translation function and an algebraic equation which shows the translation of a function?**