4

Turn Yourself Around

Rotations as Functions

MATERIALS

Compasses
Patty paper
Protractors
Rulers
Straightedges

Lesson Overview

Students analyze rotations. First they use concentric circles to rotate a triangle and determine that rotations are isometries. They then are introduced to the notation for the rotation function and use it to rotate any figure using only a protractor and ruler. Students then reverse the process and identify the center of rotation and angle of rotation given a pre-image and image of a figure. As in the previous lesson, they identify sequences of transformations to demonstrate that two figures are congruent. Students then use a graphic organizer to summarize what they have learned about translation, reflection, and rotation isometries.

Geometry

Coordinate and Transformational Geometry

(3) The student uses the process skills to generate and describe rigid transformations (translation, reflection, and rotation) and non-rigid transformations (dilations that preserve similarity and reductions and enlargements that do not preserve similarity). The student is expected to:

(B) determine the image or pre-image of a given two-dimensional figure under a composition of rigid transformations, a composition of non-rigid transformations, and a composition of both, including dilations where the center can be any point in the plane.

(C) identify the sequence of transformations that will carry a given pre-image onto an image on and off the coordinate plane.

Logical Argument and Constructions

(5) The student uses constructions to validate conjectures about geometric figures. The student is expected to:

(B) construct congruent segments, congruent angles, a segment bisector, an angle bisector, perpendicular lines, the perpendicular bisector of a line segment, and a line parallel to a given line through a point not on a line using a compass and a straightedge.

Proof and Congruence

(6) The student uses the process skills with deductive reasoning to prove and apply theorems by using a variety of methods such as coordinate, transformational, and axiomatic and formats such as two-column, paragraph, and flow chart. The student is expected to:

(C) apply the definition of congruence, in terms of rigid transformations, to identify congruent figures and their corresponding sides and angles.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 4.D, 4.J, 5.B, 5.F, 5.G

Essential Ideas

- Rotations are isometries.
- A rotation is a function, $R_{E,t}(P) = P'$ that maps its input, a point P, to another location, P'. This movement to a new location is defined by a center of rotation, E, and a rotation angle, t.
- The center of rotation lies on the perpendicular bisector of each pair of corresponding points of a pre-image and its rotated image. For this reason, the center of rotation is the point of intersection of any two of these perpendicular bisectors.

Lesson Structure and Pacing: 3 Days

Day 1

Engage

Getting Started: Concentric Circles

Students use concentric circles to rotate a triangle. They compare the pre-image and image to determine that the rotation is an isometry.

Develop

Activity 4.1: Rotation Functions

Students are introduced to and use the appropriate notation for the rotation function. They move from rotating a figure using concentric circles to rotating a figure using only a protractor and ruler.

Day 2

Activity 4.2: Determining the Center of Rotation

Students identify the center of rotation and angle of rotation given a pre-image and image of a figure. They use the fact that the center of rotation lies on the perpendicular bisector of each segment connecting corresponding points to locate the intersection point of these perpendicular bisectors as the center of rotation. Students use a protractor to determine the angle of rotation.

Day 3

Activity 4.3: Sequences of Isometries

Students use what they have learned throughout the topic to determine the sequence of translations, reflections, and rotations that take a pre-image onto its congruent image.

Demonstrate

Talk the Talk: That's What It's All About!

Students use a graphic organizer to summarize what they have learned about translation, reflection, and rotation isometries.

ENGAGE

Getting Started: Concentric Circles

Facilitation Notes

In this activity, students use concentric circles to rotate a triangle. They compare the pre-image and image to determine that the rotation is an isometry.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

As students work, look for

- The use of a protractor to create all three angles of rotation.
- The use of construction tools to duplicate the first angle of rotation to create the other two angles of rotation.

Misconceptions

- Students may assume that concentric circles must be evenly spaced. Concentric circles are circles with a common center point. The length of the radii of each concentric circle is irrelevant to the definition.
- Students may think that each vertex of the original triangle is also the vertex of each angle of rotation. Discuss the reason why each angle of rotation has point E as its vertex, and assist students in correcting their error.

Differentiation strategies

To scaffold support when creating angles of rotation,

- Model how to create an angle of rotation using a protractor for $\angle AEA'$. Then have students create $\angle BEB'$ and $\angle CEC'$ independently.
- Provide students an enlarged version of the concentric circles to complete Question 1.
- Suggest students use a different color pencil for each angle of rotation.
- Have students use dotted line segments for the angles of rotation and solid line segments as they connect the sides of $\triangle A'B'C'$.

Questions to ask

- Which circle did you use to locate point A? Point B? Point C? Does it make a difference?
- How many degrees is your rotation angle?
- Should your angle be created in a clockwise or counterclockwise direction? How do you know?
- How did you create $\angle AEA'$?
- Why is the point E important?

Have students work with a partner or in a group to complete Questions 2 and 3. Share responses as a class.

Questions to ask

- How do you know that $\overline{EA}$ and $\overline{E'A'}$ are the same length without measuring?
- Is $\triangle ABC$ congruent to $\triangle A'B'C'$? How do you know?
- Did this transformation preserve size and shape? Distances and angle measures?
- If the triangles are congruent, is this enough information to determine that rotations, like translations and reflections, are also isometries?

Differentiation strategy

To extend the activity, ask students to rotate a triangle without the use of concentric circles. This process will be explained through a Worked Example in the next activity.

Summary

Rotations are isometries that involve a common point of rotation and a common angle of rotation.

Activity 4.1
Rotation Functions

DEVELOP

Facilitation Notes

Students are introduced to and use the appropriate notation for the rotation function. They move from rotating a figure using concentric circles to rotating a figure using only a protractor and ruler.

Ask a student to read the introduction and definition aloud. Discuss as a class.

As students work, look for

Confusion caused by the use of R for both the rotation and reflection function.

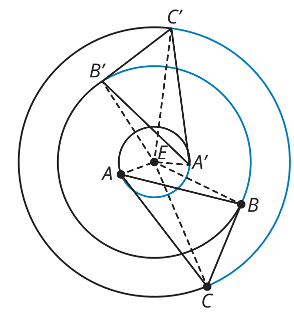
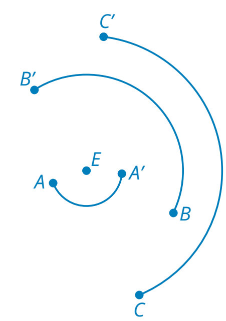
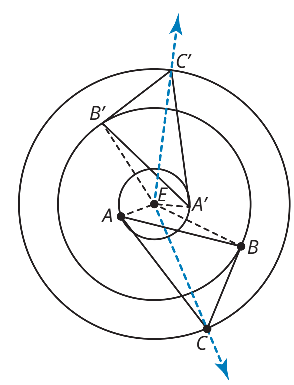
Questions to ask

- What information is needed in order to perform a rotation?
- What do the subscripts in the notation represent?
- How can you tell the notation for a reflection and rotation apart?
- How would you interpret the rotation function $R_{N,28}(\triangle PQT)$?

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

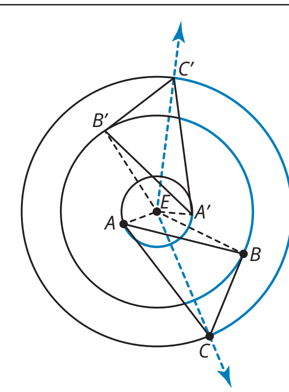
Differentiation strategy

To assist all students, guide them as to how to use their rotation diagrams from the *Getting Started* to make sense of Question 1, part (c).

<table>
<tr><td>Use a highlighter to trace the path along the circle containing point A as point A is rotated to point A′. Repeat the process for Points B and C.</td><td>
</td></tr>
<tr><td>Discuss the fact that each point did not move the same distance.</td><td></td></tr>
<tr><td>Use patty paper to trace Point E and the three highlighted arcs in their exact locations.</td><td>
</td></tr>
<tr><td>Highlight ∠CEC′ and extend it beyond the circle.</td><td>
</td></tr>
<tr><td>Place the patty paper over the circle, aligning point E.

Rotate the patty paper so that the highlighted arc from C to C′ on the patty paper is aligned with that arc in the diagram.

Then, rotate the patty paper so that the arc from A to A′ is aligned between the highlighted rays of ∠CEC′.

Repeat with the arc from B to B′.</td><td>
</td></tr>
<tr><td>Discuss the fact that the arcs are all the same angle measure because they all extend exactly from one ray of ∠CEC′ to the other.</td><td></td></tr>
</table>

Questions to ask

- What components of the rotation should be included in the subscript?
- What part of the subscript should everyone have alike? What should it be?
- What should be included in the parentheses? Is there another way this could be expressed?
- Does this rotation apply to the vertices only?
- Is every point of $\triangle ABC$ rotated about point E the same number of degrees?
- How could you demonstrate that this rotation applies to other points besides the vertices of the triangle?
- Consider the distance as point A is swept across its circle to the location of point A'. Is this the same distance for points B and B'? for C and C'?
- What is the same as each point is swept across its circle?
- How many equality statements can you write? Why is this the case?
- What is the difference between equality statements and congruent statements?

Have students work with a partner or in a group to analyze the Worked Example and complete Question 3. Share responses as a class.

Differentiation strategy

To scaffold support, suggest that besides analyzing the steps of the Worked Example, students reenact the steps alongside the directions.

Questions to ask

- Do you prefer the method demonstrated in the Worked Example or the use of concentric circles to apply the rotation function? Why?
- How do you interpret the rotation function $R_{E,75}(\overline{VH})$?
- What does the subscript E represent? What does the subscript 75 represent?
- Is $\overline{VH}$ rotated 75° clockwise or 75° counter-clockwise? How do you know?
- Did you connect points E and V first or points E and H first? Does it matter? Is one way easier than the other?
- How is Question 3, part (a) different than Question 3, part (b)?
- When completing a clockwise rotation, is it easier to first connect one point rather than the other to the point of rotation? If so, explain why.
- The length of the rays of an angle have no impact of the size of the angle. Why do the lengths of each side of the angle matter for these questions?

Summary

A rotation is a function, $R_{E,t}(P) = P'$, that maps its input, a point P, to another location, $f(P)$. This movement to a new location is defined by a center of rotation, E, and a rotation angle, t.

Activity 4.2

Determining the Center of Rotation

Facilitation Notes

In this activity, students identify the center of rotation and angle of rotation given a pre-image and image of a figure. They use the fact that the center of rotation lies on the perpendicular bisector of each segment connecting corresponding points to locate the intersection point of these perpendicular bisectors as the center of rotation. Students use a protractor to determine the angle of rotation.

Have students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Questions to ask

- Do any two points always lie on the same circle?
- Is it possible to draw two points that do not lie on the same circle?
- Given two points, how would you locate the center of a circle that contains both points on the circle?
- If two points are connected to form a line segment, will the midpoint of the line segment always lie on the perpendicular bisector of the segment?
- Will the perpendicular bisector of the segment help to determine the location of the center point of a circle that contains the endpoints of the segment? How?
- How can four points that lie on the same circle be used to locate the center point of the circle?
- Why does the intersection of two or more perpendicular bisectors of segments formed by connecting points that lie on a circle always determine the center point of the circle?
- Do the perpendicular bisectors of segments AA' and BB' intersect at the center point of the circle the points A and A' lie on? Is this also the center point of a circle the points B and B' lie on? Why?
- Is this center point of the circle also the center of rotation?

As students work, look for

Incorrect language used to describe the locations of points with respect to a circle. Points have one of three possible locations: points can lie *on* the circle, points can lie *on the interior* of a circle, or points can lie *on the exterior* of a circle. To say points lie on the circumference of a circle is incorrect because the circumference is the distance around a circle, not a region or location on a circle.

Differentiation strategy

To assist all students, have the students draw a large arc (greater than 180°) on a piece of paper. Locate any two points on the arc and connect them to form a line segment. Locate two additional points on the arc and connect them to form a second line segment. Construct the perpendicular bisector of each line segment. The intersection of the two perpendicular bisectors will determine the location of the center point of the circle that contains the large arc. Ask the students to explain why this happens.

Summary

When a center of rotation is not given, it can be determined as the intersection point of the perpendicular bisectors of the segments connecting the pre-image points and image points of a rotated figure.

Activity 4.3

Sequences of Isometries

Facilitation Notes

In this activity, students use what they have learned throughout the topic to determine the sequence of translations, reflections, and rotations which take a pre-image onto its congruent image.

Have students work with a partner or in a group to complete Question 1. Share responses as a class.

Differentiation strategies

To scaffold support,

- Suggest students use colored pencils to outline corresponding sides. This may help them to recognize the types of isometries that may be required to move the pre-image to the image.
- Recommend students use patty paper to identify or check the accuracy of their sequence of isometries.

Questions to ask

- Why does using isometries to move the pre-image to the image demonstrate the figures are congruent?
- What about the orientation of the triangles suggests that reflections may be necessary to demonstrate the figures are congruent?
- Which reflection should be performed first? Does it make a difference?
- Is there another sequence of isometries that would move the pre-image to the image? If so, describe the sequence of moves.
- What about the orientation of the trapezoids suggests that a translation and reflection may be necessary to demonstrate that the figures are congruent?
- Which isometry should be performed first? Does it make a difference?
- Is there another sequence of isometries that would move the pre-image to the image? If so, describe the sequence of moves.

Differentiation strategy

To extend the activity, have students investigate whether a single rotation or a sequence of isometries that include a rotation would move the pre-image trapezoid onto the image trapezoid.

Summary

A sequence of isometries can be used to show that two plane figures are congruent.

DEMONSTRATE

Talk the Talk: That's What It's All About!

Facilitation Notes

In this activity, students use a graphic organizer to summarize what they have learned about translation, reflection, and rotation isometries.

Have students work with a partner or in a group to complete the graphic organizers. Share responses as a class.

Differentiation strategy

As an alternative grouping method, assign each group of students one of the three graphic organizers to complete. Have each group present their information and demonstrate their transformation to the class while the rest of the class completes the graphic organizer based on the group's presentation. Display the posters for reference.

Questions to ask

- What does a translation do to a figure?
- Are all translations vertical or horizontal?
- What information is needed to write a transformation function and complete a transformation?
- Why is a directed line segment important when completing a transformation?
- What does a reflection do to a figure?
- Describe the different ways a figure can be reflected. Are all reflections vertical or horizontal?
- What information is needed to write a reflection function and complete a reflection?
- How is the line of reflection determined?
- What do perpendicular line segments have to do with reflections?
- What is the relationship between a line of reflection and the segment formed by corresponding points of the pre-image and image?
- What is the relationship between the distances of corresponding points from the line of reflection?
- What does a rotation do to a figure?
- Describe the different ways a figure can be rotated.
- What information is needed to write a rotation function and complete a rotation?
- How is a center of rotation determined?
- What do perpendicular line segments have to do with rotations?
- What do translations, reflections, and rotations have in common?
- What is meant be a rigid motion?
- What is meant by a transformation?
- What is an example of a transformation that is not a rigid motion?
- What isometry is most interesting to you? Why?
- What isometry is most challenging to you? Why?
- Do you prefer completing isometries with or without a coordinate plane? Why?

Summary

The rigid motion transformations are isometries which preserve size and shape. Translations require a directed line segment, reflections require a line of reflection, and rotations require a center of rotation and angle of rotation.

NOTES

Warm Up Answers

1. 180°
2. 90°
3. 90°

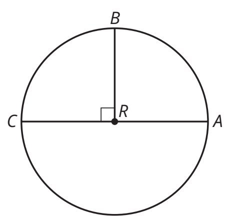
4

Turn Yourself Around

Rotations as Functions

Warm Up

Identify each central angle measure.



1. m∠ARC
2. m∠ARB
3. m∠CRB

Learning Goals

- Represent rotations in the plane.
- Describe rotation transformations as functions that take points as inputs and output rotated points.
- Specify a sequence of translations, reflections, and rotations that will carry a figure onto a congruent figure.

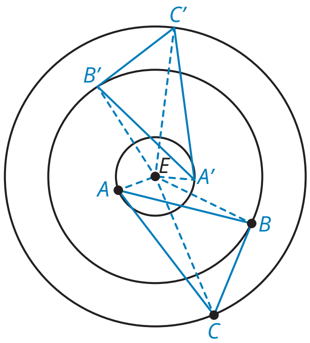
Key Term

- rotation

You have studied circles and rotations. How can you use circles to define and describe rotations on the plane?

Answers

1. Sample answer.
   A 150° rotation of $\triangle ABC$ to form $\triangle A'B'C'$



2. The line segments are congruent because they are each radii of the same circle.
3. The transformation is an isometry because $\triangle ABC \cong \triangle A'B'C'$. The triangles are congruent because the corresponding sides and angles of the triangles are congruent.

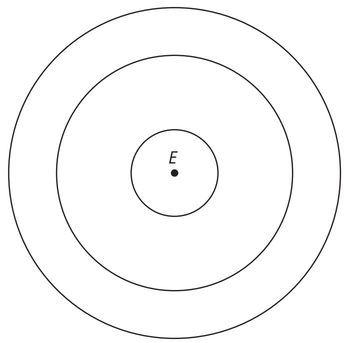
GETTING STARTED

Concentric Circles

Concentric circles are circles with a common center point.

You have constructed circles and identified parts of circles in previous lessons. Circles are important for understanding rotations.

1. Consider three concentric circles with center at point *E*.



a. Draw one point on each circle so that you can connect the points to form $\triangle ABC$. Draw the sides of the triangle.

b. Choose a positive or negative rotation angle, *t*.

c. Draw *A′* as an endpoint of *EA′* such that $\angle AEA'$ has a measure of *t* degrees.

d. Repeat part (c) to draw points *B′* and *C′*. Connect the transformed points to form $\triangle A'B'C'$.

Remember:

An isometry preserves distances and angle measures.

2. What do you notice about the pairs of line segments $\overline{EA}$ and $\overline{EA'}$, $\overline{EB}$ and $\overline{EB'}$, and $\overline{EC}$ and $\overline{EC'}$? Explain your observations.

3. Is the transformation you created an isometry? Explain your thinking.

ACTIVITY
4.1

Rotation Functions

A **rotation** is a function that maps its input, a point P, to another location, $f(P)$. This movement to a new location is defined by a center of rotation, E, and a rotation angle, t. For this reason, a rotation function is written as $R_{E,t}(P)$.

Because the rotation is defined about a point E, the movement of a specific point traces an arc that is part of the circumference of a circle with center E. The arc can be labeled by the starting point, P, and the endpoint, P', or as, $\overset{\frown}{PP'}$. The degree measure of this arc is equivalent to the degree rotation, t, that creates a central angle in Circle E.

Remember:

A circle is named by the point that is its center.

1. Consider the rotation you created in the previous activity.

a. Write a function of the form $R_{E,t}(\triangle ABC)$ to describe the rotation of $\triangle ABC$.

b. Explain how your function represents the rotation of every point of $\triangle ABC$.

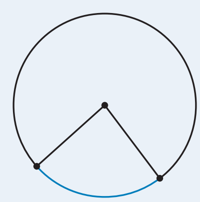
c. Do the arcs you created all have the same measure? Explain your answer.

2. Write equality and congruence statements to compare the corresponding sides of the pre-image and the image.

Think about:

A circle is a rotation of a point around a given center 360 degrees.

A circle is a locus of points that are all a given distance from a center point.



An angle is the measure of the distance the point is rotated as measured by the central angle.

Answers

1a. Sample answer.
$R_{E,\,150}(\triangle ABC) = \triangle A'B'C'$

1b. When the vertices are connected to make the sides of the triangle, every point of each side is rotated $t°$ about point E.

1c. Sample answer.
All points do not travel the same distance as the triangle is rotated; however, all points travel along an arc of the same degree measure as each of them move around their respective concentric circle.

2. $AB = A'B'$
$BC = B'C'$
$CA = C'A'$
$\overline{AB} \cong \overline{A'B'}$
$\overline{BC} \cong \overline{B'C'}$
$\overline{CA} \cong \overline{C'A'}$

ELL Tip

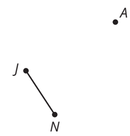
Assess students' prior knowledge of the term *locus*. Explain that a *locus* can be *a particular point, position, or place*, or in a mathematical context, it is *the set of all points whose location is determined by certain conditions*. Read aloud the second sentence in the "Think about" column in the activity, "We say that a circle is a *locus* of points that are all a given distance from a center point." Sketch a circle with a given center and identify several points on the circle, noting that they are all the same distance from the center. Clarify any remaining misunderstandings about the application of *locus* in regards to a circle.

It is possible to apply the rotation function to a figure by using a protractor and ruler.

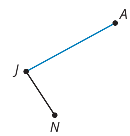
Worked Example

$R_{A,40}(\overline{JN})$

This means to rotate $\overline{JN}$ 40°, using point A as the center of rotation.

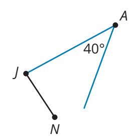


1. Draw a line segment from the center of rotation, A, to one endpoint of the line segment.

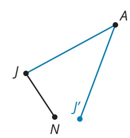


2. Using a protractor, draw a 40° angle. Use the center of rotation, A, as the vertex and the line segment drawn, $\overline{AJ}$, as one side of the angle.

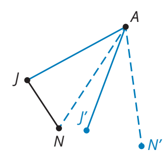
   Since the angle measure is positive, place the angle in the counter-clockwise direction of the line segment drawn.



3. Use a ruler or compass to extend the side of the angle so that it is the same length as $\overline{AJ}$. Label the other endpoint J'.



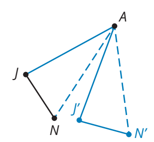
4. Repeat steps 1, 2, and 3 using the other endpoint of the original line segment.



5. Connect endpoints J' and N'.

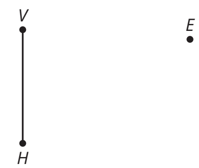
   $R_{A,40}(\overline{JN}) = \overline{J'N'}$

   Segment $\overline{J'N'}$ is the result of a 40° rotation of $\overline{JN}$ about point A.

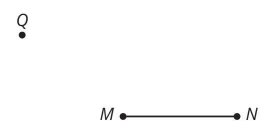


3. Complete each rotation using the given function.

a. $R_{E,75}(\overline{VH})$



b. $R_{Q,-30}(\overline{MN})$

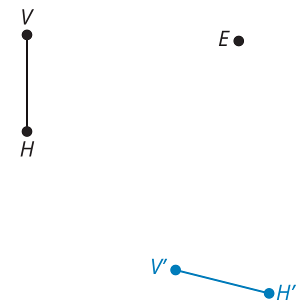


Remember:

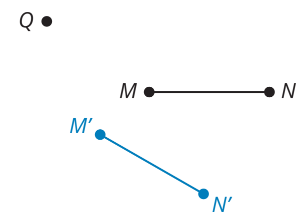
When you construct, you use only a compass and straightedge. Here you are drawing, so you can use those tools as well as measuring tools, such as a protractor and ruler.

Answers

3a.

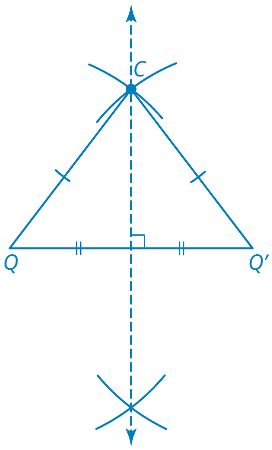


3b.



Answers

1. Sample answer.



2a. See transformation below.

2b. See transformation below.

ACTIVITY **4.2**

Determining the Center of Rotation

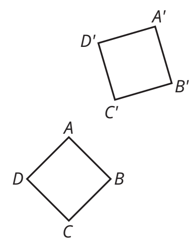
You have seen that corresponding points on rotated figures are equidistant from the center of rotation.

1. Draw an example to explain why Tori is correct.

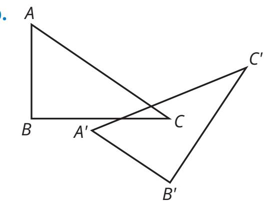
> Tori
> If two points Q and Q′ are equidistant from the center, then the perpendicular bisector of $\overline{QQ'}$ passes through the center.

2. Use what you know to determine the center of rotation and rotation angle for the transformation of each figure. Write each rotation as a function.

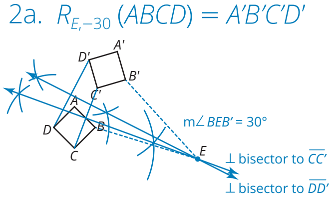
a.



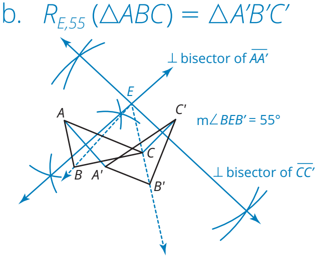
b.



2a. $R_{E,-30}(ABCD) = A'B'C'D'$



2b. $R_{E,55}(\triangle ABC) = \triangle A'B'C'$



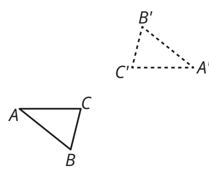
ACTIVITY
4.3

Sequences of Isometries

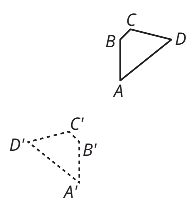
You have learned that translations, reflections, and rotations are isometries, which means that they preserve distances and angle measures.

1. Describe and sketch the sequence of isometries that shows that the two figures in each pair are congruent. Images are shown with dashed lines.

a.



b.



Answers

1a. Check students' sketches.
Sample answer. Triangle *ABC* can be reflected across a horizontal line and then reflected across a vertical line to demonstrate the figures are congruent.

1b. Check students' sketches.
Sample answer. Quadrilateral *ABCD* can be reflected across a vertical line and then translated down to demonstrate the figures are congruent.

Answers

1a-c. See answers at the end of this lesson.

NOTES

TALK the TALK

That's What It's All About!

1. Complete the graphic organizers to summarize the characteristics of each.

   a. Translation
   b. Reflection
   c. Rotation

Graphic Organizer

Geometric Object

Function Notation

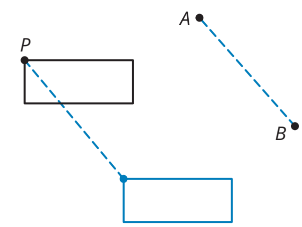
Translation

Example

Description

Answer

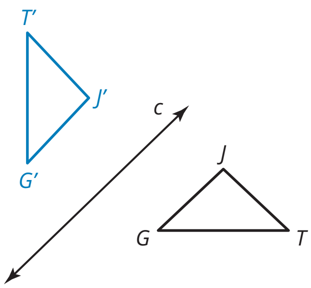
1a. Sample answers.
Graphic Organizer: Translation
Geometric Object: Any figure in a plane, such as a point, line, angle, or polygon.
Function Notation: $T_{AB}(P) = P'$ where $\overline{AB}$ is a directed line segment.
Example: $T_{AB}(P) = P'$



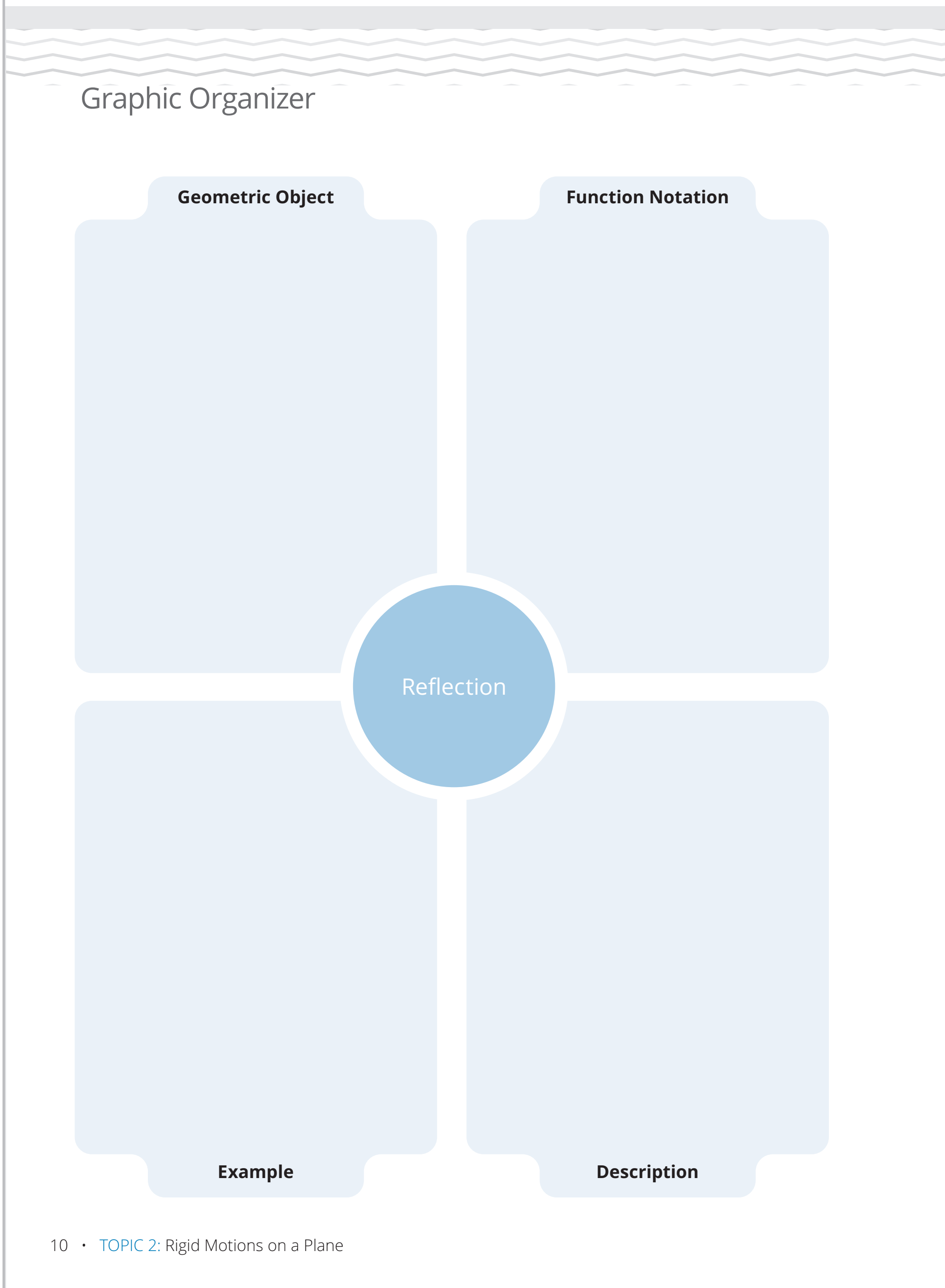
Description: Each point in the pre-image is translated the distance and direction of $\overline{AB}$ to create the image.

Answer

1b. Sample answers.
Graphic Organizer: Reflection
Geometric Object: Any figure in a plane, such as a point, line, angle, or polygon.
Function Notation: $R_\ell(P) = P'$ where ℓ is a line of reflection.
Example: $R_c(\triangle GJT) = \triangle G'J'T'$



Description: Each point in the pre-image is reflected to the opposite location with respect to the line of reflection. The line segment formed by corresponding points is perpendicular to the line of reflection. Corresponding points are equidistant from the line of reflection.


Graphic Organizer

Geometric Object

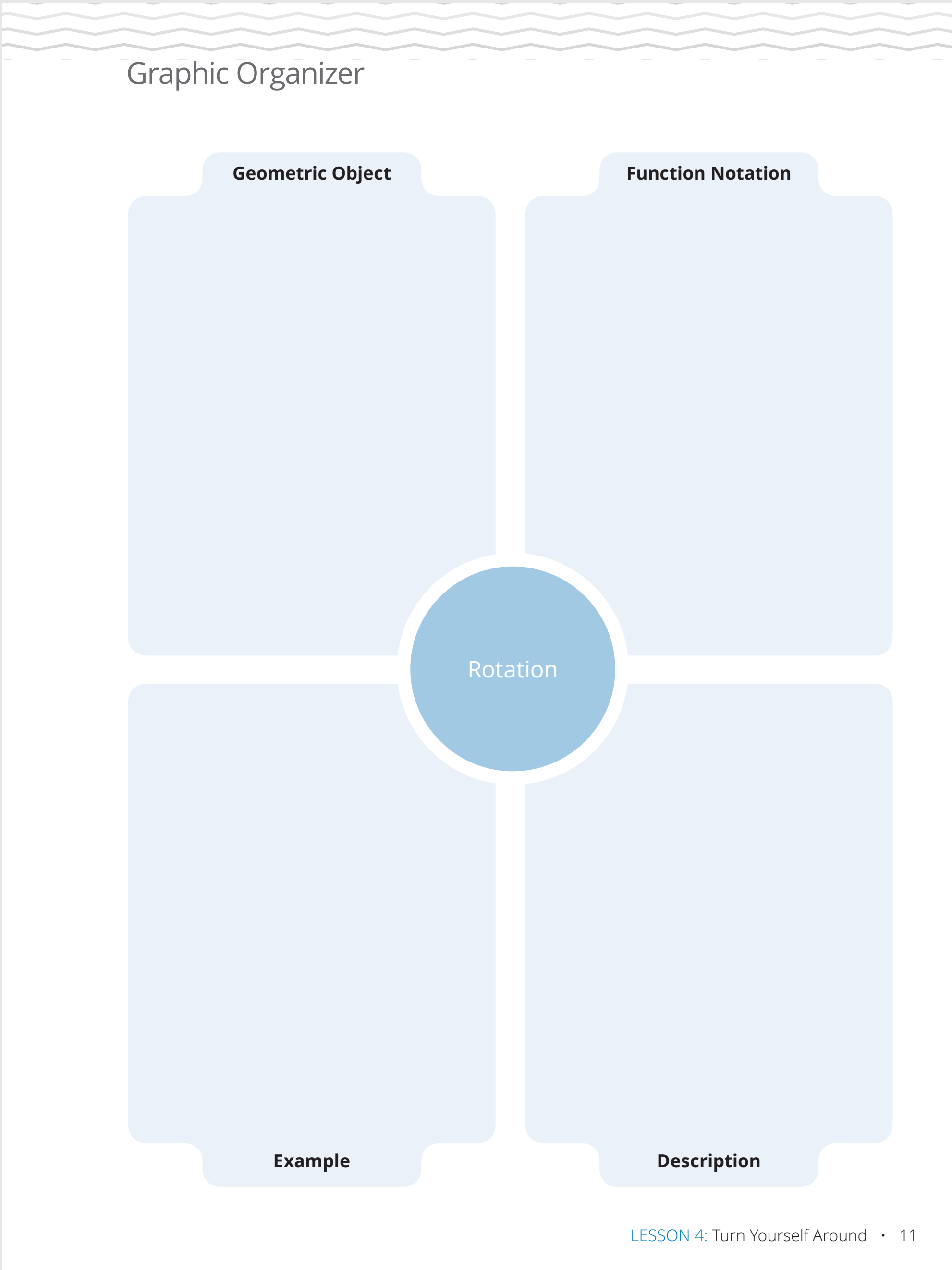
Function Notation

Reflection

Example

Description

10 • TOPIC 2: Rigid Motions on a Plane



Graphic Organizer

Geometric Object

Function Notation

Rotation

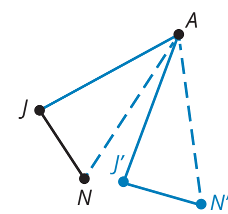
Example

Description

LESSON 4: Turn Yourself Around • 11

Answer

1c. Sample answers.
Graphic Organizer: Rotation
Geometric Object: Any figure in a plane, such as a point, line, angle, or polygon.
Function Notation: $R_{E,t}(P) = P'$ where E is the center of rotation and t is the angle of rotation.
Example: $R_{A,40}(\overline{JN}) = (\overline{J'N'})$



Description: Each point of $\overline{JN}$ is rotated 40° counterclockwise around point A to create $\overline{J'N'}$.