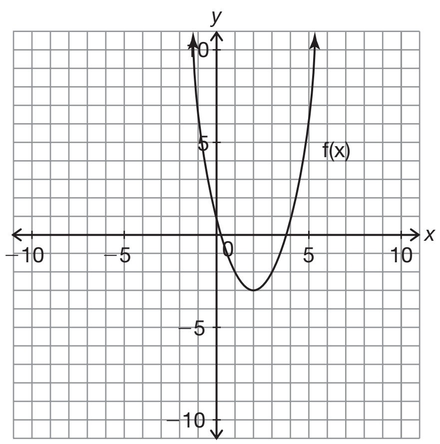
5

Slide, Flip, Turn: The Latest Dance Craze?

Translations, Rotations, and Reflections on the Coordinate Plane

Warm Up

Consider the graph of $f(x)$.



Complete the table.

$f(x)$	$-f(x)$	$f(-x)$
(−1, 6)		
(0, 1)		
(1, −2)		

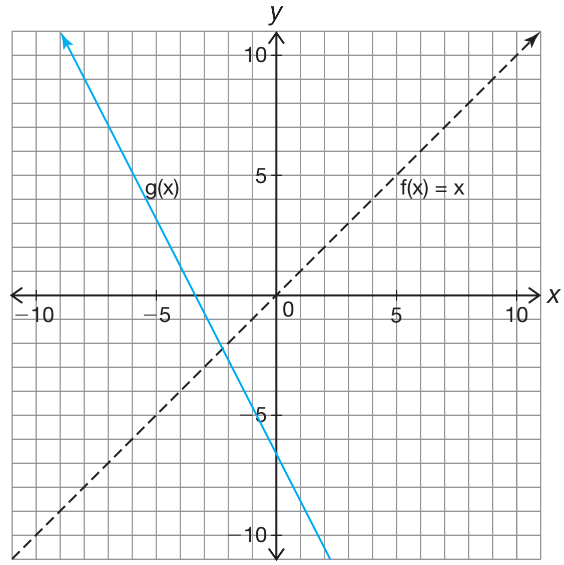
Learning Goals

- Translate geometric figures on a coordinate plane.
- Rotate geometric figures on a coordinate plane.
- Reflect geometric figures on a coordinate plane.

You have learned that translations, reflections, and rotations behave like functions. What correlation can be drawn between transforming the points that make up a figure and the points that make up the graph of a function?

Basic Steps

Consider the basic function $f(x) = x$ and the transformed graph $g(x)$.



1. Describe a sequence of transformations from the graph of $f(x)$ to the graph of $g(x)$.

2. Write the function equation for $g(x)$ in terms of $f(x)$.

3. Complete the table with the corresponding coordinates.

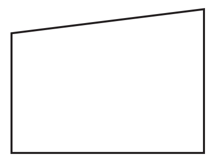
$f(x)$	$g(x)$
(−1, −1)	
(0, 0)	
(1, 1)	
(2, 2)	

ACTIVITY

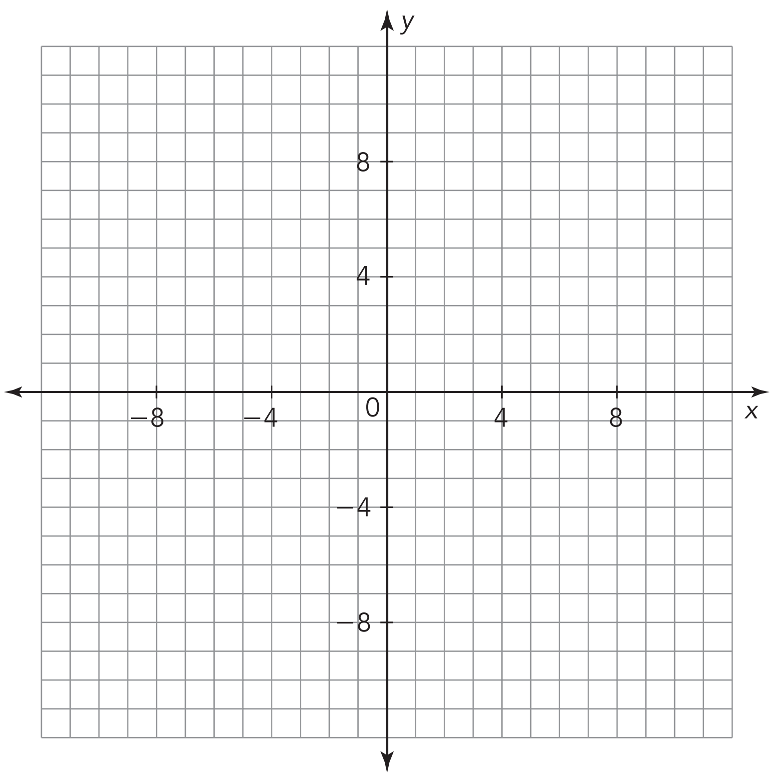
5.1

Translating Geometric Figures on the Coordinate Plane

Cut out a copy of the figure shown.



1. Graph trapezoid *ABCD* by plotting the points *A* (3, 9), *B* (3, 4), *C* (11, 4), and *D* (11, 10).



2. Translate trapezoid *ABCD* on the coordinate plane. Graph the image and record the vertex coordinates in the table.

a. Translate trapezoid *ABCD* 15 units to the left to form trapezoid *A′B′C′D′*.

b. Translate trapezoid *ABCD* 12 units down to form trapezoid *A″B″C″D″*.

Coordinates of Trapezoid *ABCD*	Coordinates of Trapezoid *A′B′C′D′*	Coordinates of Trapezoid *A″B″C″D″*
A (3, 9)		
B (3, 4)		
C (11, 4)		
D (11, 10)		
Point on Trapezoid *ABCD* (*x*, *y*)		

Let's consider translations without graphing.

3. The vertices of parallelogram *DEFG* are *D* (−9, 7), *E* (−12, 2), *F* (−3, 2), and *G* (0, 7).

a. Determine the vertex coordinates of image *D′E′F′G′* if parallelogram *DEFG* is translated 14 units down.

b. How did you determine the image coordinates without graphing?

c. Determine the vertex coordinates of image *D″E″F″G″* if parallelogram *DEFG* is translated 8 units to the right.

d. How did you determine the image coordinates without graphing?

ACTIVITY 5.2

Rotating Geometric Figures on the Coordinate Plane

Recall that a rotation is a rigid motion that turns a figure about a fixed point, called the point of rotation. The figure is rotated in a given direction for a given angle, called the angle of rotation. The angle of rotation is the measure of the amount the figure is rotated about the point of rotation. The direction of a rotation can either be clockwise or counterclockwise.

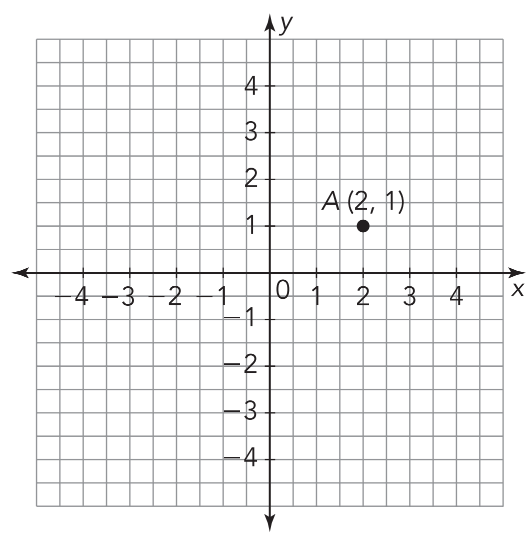
Let's rotate point *A* about the origin. The origin will be the point of rotation, and you will rotate point *A* 90°, 180°, and 270°.

First, let's rotate point *A* 90° counterclockwise about the origin.

Worked Example

Step 1: Plot a point anywhere in the first quadrant, but *not* at the origin.

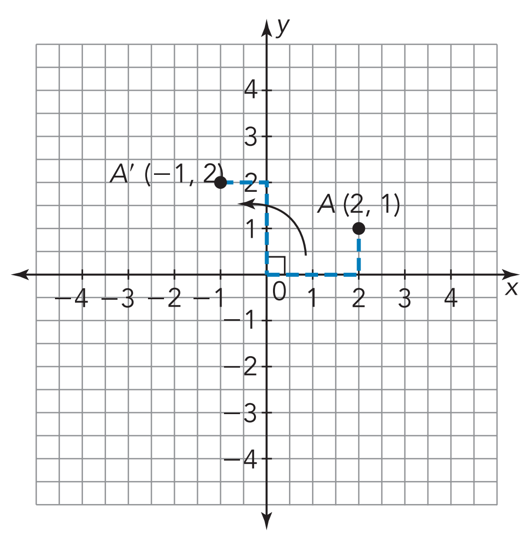
Point *A* is plotted at (2, 1).



Step 2: Next, draw a "hook" from the origin to point *A*, using the coordinates and horizontal and vertical line segments as shown.

Step 3: Rotate the "hook" 90° counterclockwise as shown.

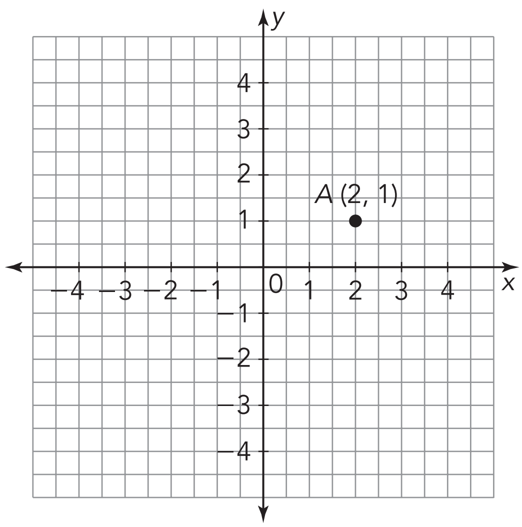
Point *A*′ is located at (–1, 2). Point *A* has been rotated 90° counterclockwise about the origin.



1. What do you notice about the coordinates of point *A* and the coordinates of point *A′*?

2. Predict what the coordinates of point *A″* will be if you rotate point *A′* 90° counterclockwise about the origin.

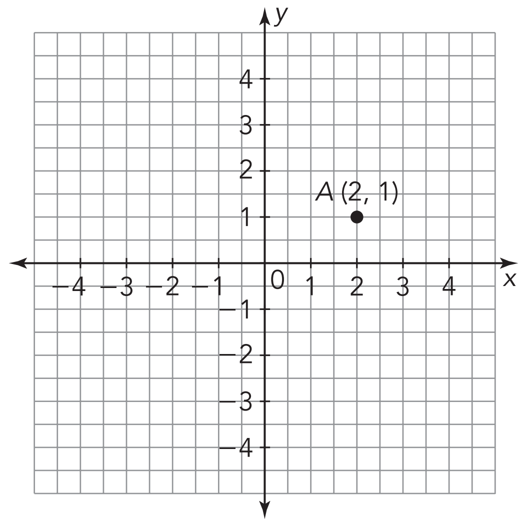
3. Graph and label point *A′*. Rotate point *A′* about the origin 90° counterclockwise on the coordinate plane shown. Label the point *A″*.



a. What are the coordinates of point *A″*? Was your prediction for the coordinates of point *A″* correct?

b. What do you notice about the coordinates of points *A* and *A″*? How are the two points related?

4. Graph and label points *A′* and *A″*. Rotate point *A″* about the origin 90° counterclockwise on the coordinate plane shown. Label the point *A‴*.



a. What are the coordinates of point *A‴*?

b. What do you notice about the coordinates of point *A* and point *A‴*? How are the two points related?

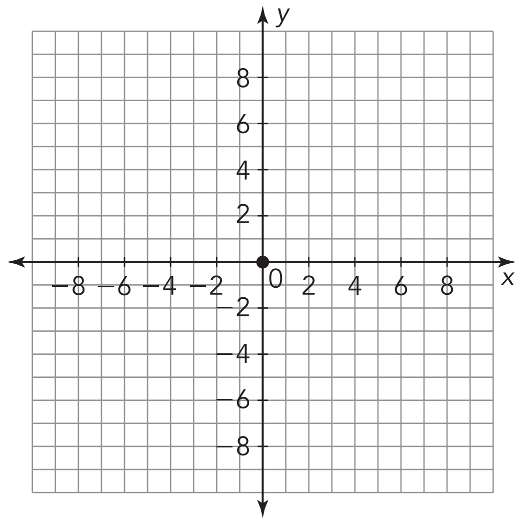
You may have noticed that the values of the *x*- and *y*-coordinates seem to switch places for every 90° rotation about the origin. You may have also noticed that the rotation from point *A* to *A″* is a 180° counterclockwise rotation about the origin, and that the rotation from point *A* to *A‴* is a 270° counterclockwise rotation about the origin.

5. Determine the coordinates of point (*x*, *y*) after rotations of 90°, 180°, 270°, and 360°.

Original Point	Coordinates After a 90° Counterclockwise Rotation About the Origin	Coordinates After a 180° Counterclockwise Rotation About the Origin	Coordinates After a 270° Counterclockwise Rotation About the Origin	Coordinates After a 360° Counterclockwise Rotation About the Origin
(*x*, *y*)				

You can verify that the information in the table is correct by using a test point. Plot test point *Q* on the coordinate plane and rotate the point 90°, 180°, 270°, and 360° counterclockwise about the origin.

6. Graph and label point *Q* at (5, 7) on the coordinate plane.



Remember:

Remember that the table shows values for the coordinates, but coordinates for a plotted point are always in the form (*x*, *y*)!

7. Use the origin (0, 0) as the point of rotation.

a. Rotate the pre-image *Q* 90° counterclockwise about the origin. Label the image *Q′*. Determine the coordinates of image *Q′*, then describe how you determined the location of point *Q′*.

b. Rotate the pre-image *Q* 180° counterclockwise about the origin. Label the image *Q″*. Determine the coordinates of image *Q″*, then describe how you determined the location of image *Q″*.

c. Rotate the pre-image *Q* 270° counterclockwise about the origin. Label the image *Q‴*. Determine the coordinates of point *Q‴*, then describe how you determined the location of image *Q‴*.

d. Rotate the pre-image *Q* 360° counterclockwise about the origin. Label the image *Q''''*. Determine the coordinates of point *Q''''*, then describe how you determined the location of image *Q''''*.

Remember:

Make sure you have the trapezoid that you cut out earlier.

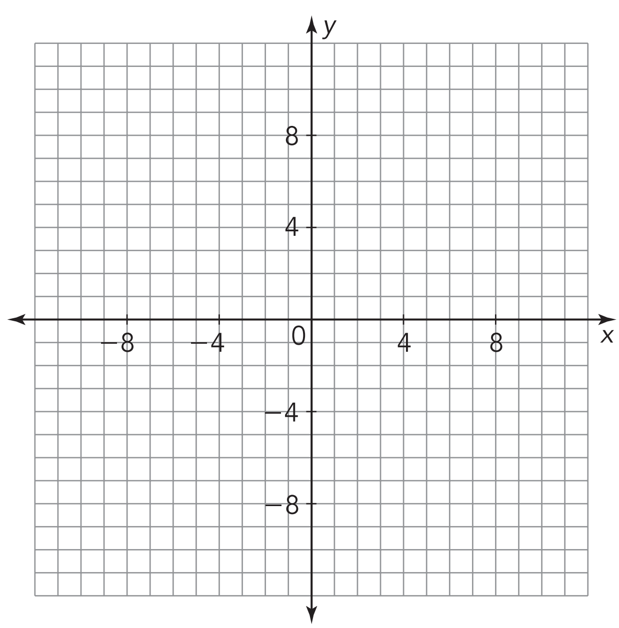
You have been rotating points about the origin on a coordinate plane. However, do you think polygons can also be rotated on the coordinate plane?

You can use models to help show that you *can* rotate polygons on a coordinate plane. However, before we start modeling the rotation of a polygon on a coordinate plane, let's graph the trapezoid to establish the pre-image.

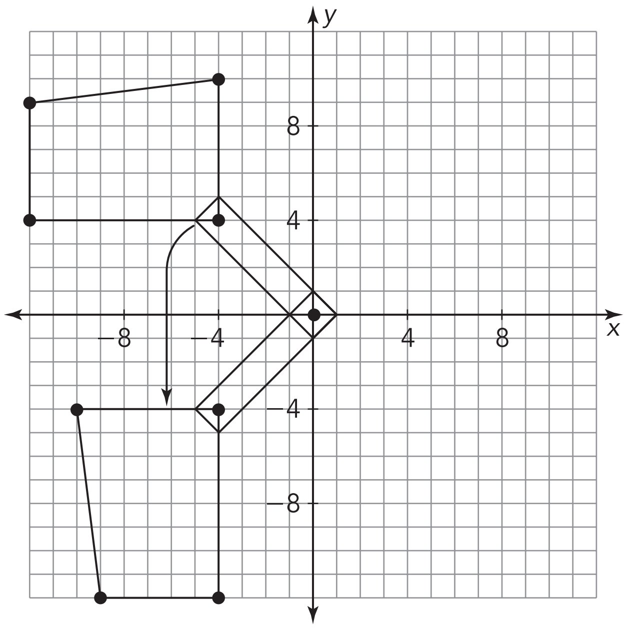
8. Graph trapezoid *ABCD* by plotting the points *A* (−12, 9), *B* (−12, 4), *C* (−4, 4), and *D* (−4, 10).

Now that you have graphed the pre-image, you are ready to model the rotation of the polygon on the coordinate plane.

- **First, fold a piece of tape in half and tape it to both sides of the trapezoid you cut out previously.**
- **Then, take your trapezoid and set it on top of trapezoid *ABCD* on the coordinate plane, making sure that the tape covers the origin (0, 0).**
- **Finally, put a pin or your pencil point through the tape at the origin and rotate your model counterclockwise.**



The 90° counterclockwise rotation of trapezoid *ABCD* about the origin is shown.



9. Rotate trapezoid *ABCD* about the origin for each given angle of rotation. Graph and label each image on the coordinate plane and record the coordinates in the table.

a. Rotate trapezoid *ABCD* 90° counterclockwise about the origin to form trapezoid *A′B′C′D′*.

b. Rotate trapezoid *ABCD* 180° counterclockwise about the origin to form trapezoid *A″B″C″D″*.

Coordinates of Trapezoid *ABCD*	Coordinates of Trapezoid *A′B′C′D′*	Coordinates of Trapezoid *A″B″C″D″*
A (−12, 9)		
B (−12, 4)		
C (−4, 4)		
D (−4, 10)		
Point on Trapezoid *ABCD*		

10. What similarities do you notice between rotating a single point about the origin and rotating a polygon about the origin?

Let's consider rotations without graphing.

11. The vertices of parallelogram *DEFG* are *D* (−9, 7), *E* (−12, 2), *F* (−3, 2), and *G* (0, 7).

a. Determine the vertex coordinates of image *D′E′F′G′* if parallelogram *DEFG* is rotated 90° counterclockwise about the origin.

b. How did you determine the image coordinates without graphing?

c. Determine the vertex coordinates of image *D″E″F″G″* if parallelogram *DEFG* is rotated 180° counterclockwise about the origin.

d. How did you determine the image coordinates without graphing?

e. **Determine the vertex coordinates of image *D'''E'''F'''G'''* if parallelogram *DEFG* is rotated 270° counterclockwise about the origin.**

f. **How did you determine the image coordinates without graphing?**

12. Dante claims that if he is trying to determine the coordinates of an image that is rotated 180° about the origin, it does not matter which direction the rotation occurred. Desmond claims that the direction is important to know when determining the image coordinates. Who is correct? Explain why the correct student's rationale is correct.

ACTIVITY 5.3 Reflecting Geometric Figures on the Coordinate Plane

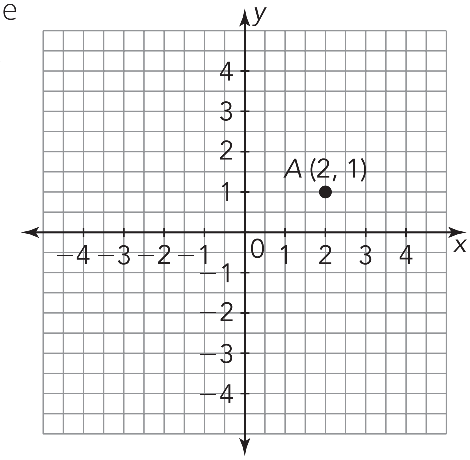
Recall that figures that are mirror images of each other are called reflections. A reflection is a rigid motion that reflects, or "flips," a figure over a given line called a line of reflection. A line of reflection is a line over which a figure is reflected so that corresponding points are the same distance from the line.

Let's reflect point *A* over the *y*-axis.

Worked Example

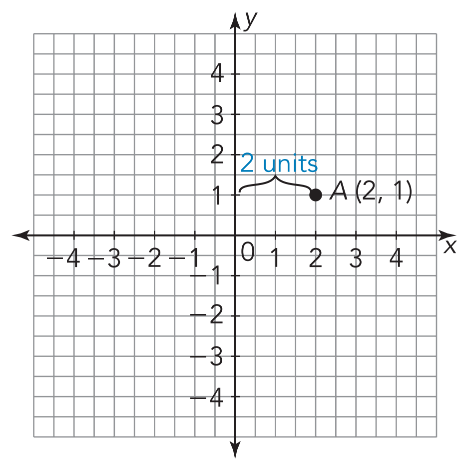
Step 1: Plot a point anywhere in the first quadrant, but *not* at the origin.

Point A is plotted at (2, 1)



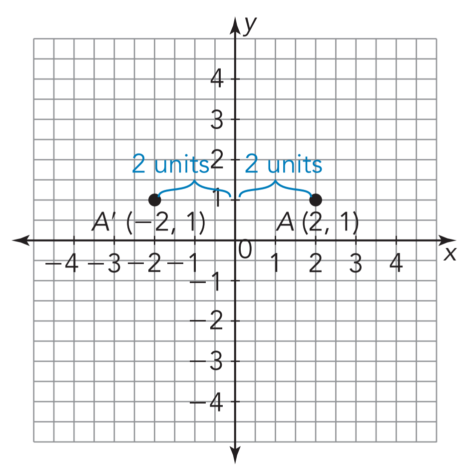
Step 2: Next, count the number of x-units from point A to the y-axis.

Point A is 2 units from the y-axis.



Step 3: Then, count the same number of x-units on the opposite side of the y-axis to locate the reflection of point A. Label the point A'.

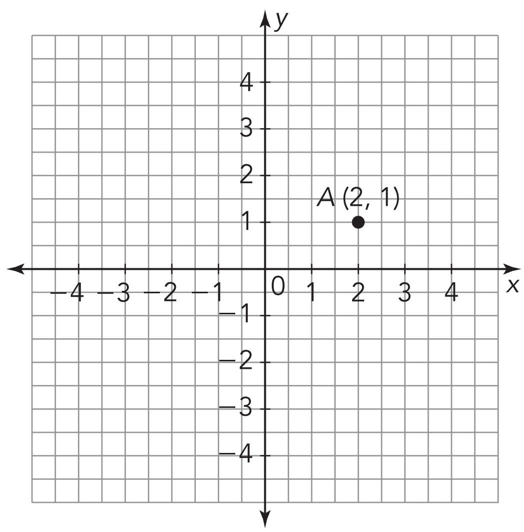
Step 4: Point A' is located at (−2, 1). Point A has been reflected over the y-axis.



1. **What do you notice about the coordinates of point *A* and the coordinates of image *A*′?**

2. **Predict the coordinates of *A*″ if point *A* is reflected over the *x*-axis. Explain your reasoning.**

3. **Reflect point *A* over the *x*-axis on the coordinate plane shown. Verify whether your prediction for the location of the image was correct. Graph the image and label it *A*″.**



4. **What do you notice about the coordinates of *A* and *A*″?**

The coordinates of a pre-image reflected over either the *x*-axis or the *y*-axis can be used to determine the coordinates of the image.

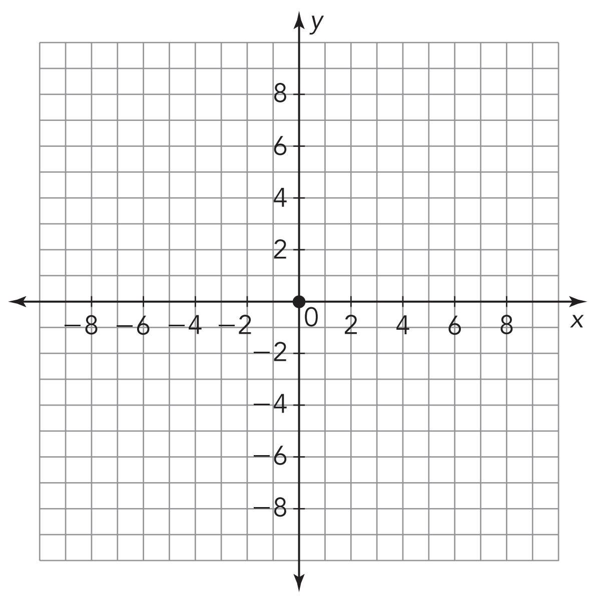
Remember:

Does this table still make sense if the line of reflection is not the *x*- or *y*-axis?

5. Determine the coordinates of point (*x*, *y*) after reflections about the *x*-axis or *y*-axis.

Original Point	Coordinates of Image After a Reflection Over the *x*-axis	Coordinates of Image After a Reflection Over the *y*-axis
(*x*, *y*)		

6. Graph point *J* at (5, 7) on the coordinate plane shown.



7. Reflect point *J* over the *y*-axis on the coordinate plane. Label the image *J′*. Determine the coordinates of *J′*. Then, describe how you determined the location of image *J′*.

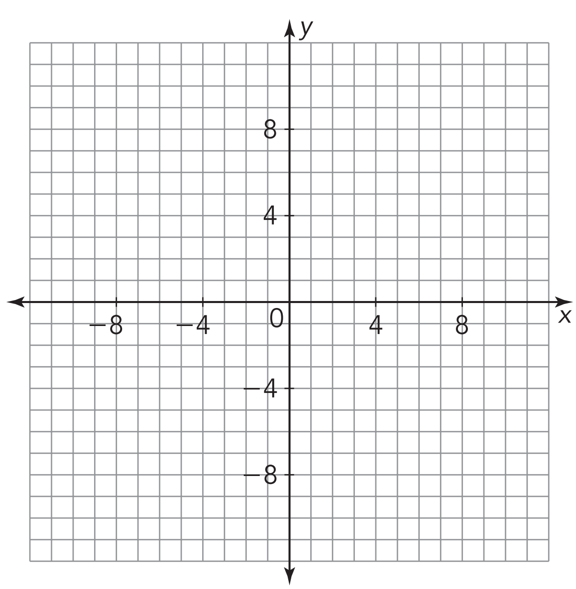
8. **Reflect point *J* over the *x*-axis on the coordinate plane. Label the image *J*″. Determine the coordinates of *J*″. Then, describe how you determined the location of image *J*″.**

You can also reflect polygons on the coordinate plane. You can model the reflection of a polygon across a line of reflection. Just as with rotating a polygon on a coordinate plane, you will first need to establish a pre-image.

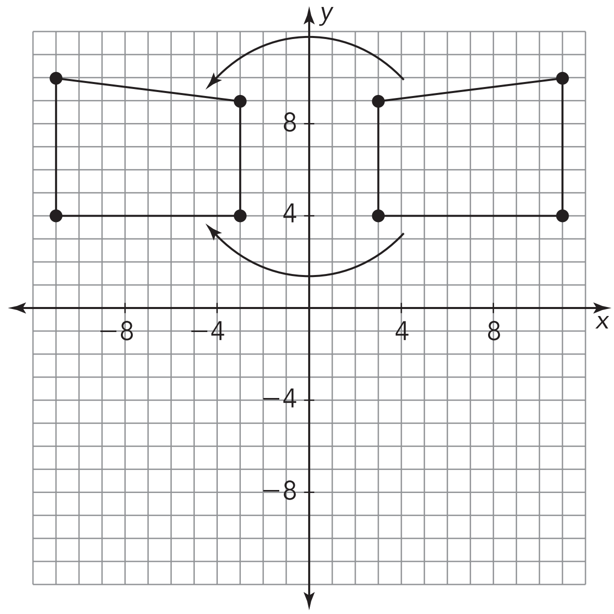
9. **Graph trapezoid *ABCD* by plotting the points *A* (3, 9), *B* (3, 4), *C* (11, 4), and *D* (11, 10).**

Now that you have graphed the pre-image, you are ready to model the reflection of the polygon on the coordinate plane. For this modeling, you will reflect the polygon over the *y*-axis.

- **First, take your trapezoid that you cut out previously and set it on top of trapezoid *ABCD* on the coordinate plane.**
- **Next, determine the number of units point *A* is from the *y*-axis.**
- **Then, count the same number of units on the opposite side of the *y*-axis to determine where to place the image in Quadrant II.**
- **Finally, physically flip the trapezoid over the *y*-axis like you are flipping a page in a book.**



The reflection of trapezoid *ABCD* over the *y*-axis is shown.



10. Reflect trapezoid *ABCD* over each given line of reflection. Graph and label each image on the coordinate plane and record each image's coordinates in the table.

a. Reflect trapezoid *ABCD* over the *y*-axis to form trapezoid *A′B′C′D′*.

b. Reflect trapezoid *ABCD* over the x-axis to form trapezoid *A″B″C″D″*.

Coordinates of Trapezoid *ABCD*	Coordinates of Trapezoid *A′B′C′D′*	Coordinates of Trapezoid *A″B″C″D″*
A (3, 9)		
B (3, 4)		
C (11, 4)		
D (11, 10)		
Point on Trapezoid *ABCD* (*x*, *y*)		

11. What similarities do you notice between reflecting a single point over the *x*- or *y*-axis and reflecting a polygon over the *x*- or *y*-axis?

Let's consider reflections without graphing.

12. The vertices of parallelogram *DEFG* are *D* (−9, 7), *E* (−12, 2), *F* (−3, 2), and *G* (0, 7).

a. Determine the vertex coordinates of image *D′E′F′G′* if parallelogram *DEFG* is reflected over the *x*-axis.

b. How did you determine the image coordinates without graphing?

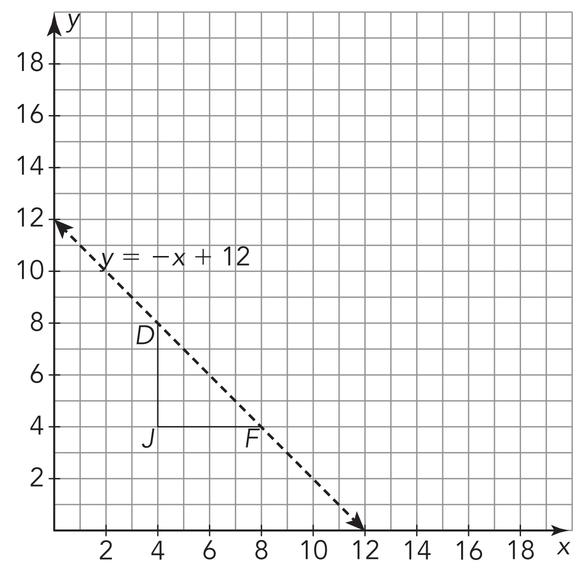
c. Determine the vertex coordinates of image *D″E″F″G″* if parallelogram *DEFG* is reflected over the *y*-axis.

d. How did you determine the image coordinates without graphing?

ACTIVITY 5.4 Multiple Transformations

You can also perform multiple transformations, or a composition of transformations, on the coordinate plane.

1. Consider ΔJDF on the coordinate plane shown.



a. Write the coordinates of the vertices of the triangle.

b. Show a reflection of ΔJDF across the line $y = -x + 12$ shown. Label the new image as $\Delta J'D'F'$. Write the coordinates of the vertices of the image.

c. Show a translation of the image to the right 10 units. Label the new image as $\Delta J''D''F''$. Write the coordinates of the vertices of the image.

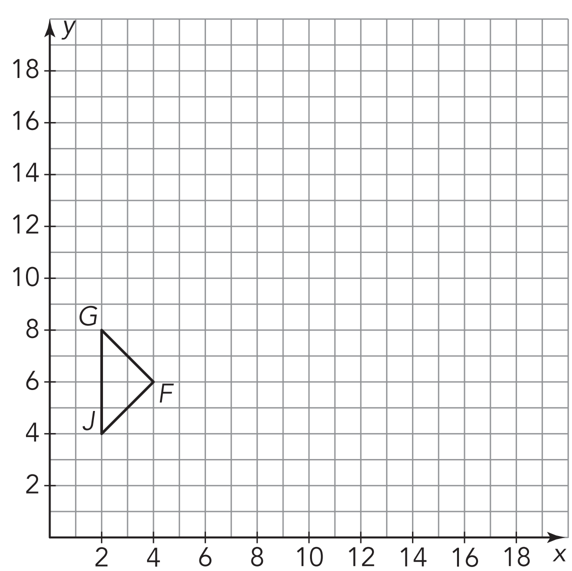
2. Bianca says that there are many different ways to use multiple transformations to transform ΔJDF to create $\Delta J''D''F''$.

Is Bianca correct? If so, identify a sequence of rigid motions that can transform ΔJDF to create $\Delta J''D''F''$.

3. Describe how you could determine the coordinates of the pre-image, ΔJDF, if you were given the coordinates of the image, $\Delta J''D''F''$, and the sequence of rigid motions.

4. Consider ΔJGF on the coordinate plane shown. Determine the coordinates of the vertices of the image, $\Delta J'G'F'$, under the composition of transformations given:

- **a rotation of 90° clockwise about point F**
- **a translation 4 units down**

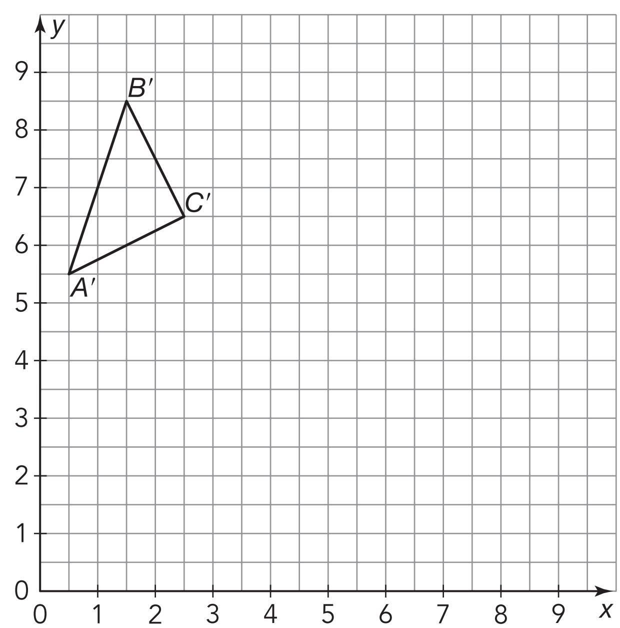


5. **Describe a single rigid motion that could produce the same result as the composition of transformations in Question 4. Explain your reasoning.**

6. **Triangle *A′B′C′* is the result of this composition of rigid motions:**

   - **a translation up 2 units**
   - **a reflection over the line $x = 2\frac{1}{2}$**

Determine the coordinates of the vertices of the pre-image, triangle *ABC*. Explain your reasoning.



NOTES

TALK the TALK

Turn Yourself Around

You know that a line is determined by two points. The slope of any line represented on a coordinate plane can be given by $\frac{y_2 - y_1}{x_2 - x_1}$.

You also now know that when rotating a point (x, y) 90° counterclockwise about the origin, the x-coordinate of the original point maps to the y-coordinate of the transformed point and the y-coordinate of the original point maps to the opposite of the x-coordinate of the transformed point.

1. Rewrite the slope ratio above to describe the slope of a line that has been rotated 90° counterclockwise. What do you notice? Explain your reasoning.

Think about:

When two lines are perpendicular to each other, how can you describe their slopes?

2. Complete the sentence using *always, sometimes,* or *never*.

Images that result from a translation, rotation, or reflection are ____________________ congruent to the original figure.