1

Elemental

Formal Reasoning in Euclidean Geometry

Warm Up

Solve for *x*.

1. $x + 4x = 90$

2. $0.5x + 2x = 90$

3. $x + 2x = 180$

4. $0.5x + 4x = 180$

Learning Goals

- Identify the hypothesis and conclusion of a conditional statement.
- Explore the truth values of conditional statements.
- Use a truth table.
- Differentiate between postulates and theorems.
- Differentiate between Euclidean and spherical geometry.

Key Terms

- counterexample
- truth value
- truth table
- Euclidean geometry
- Linear Pair Postulate
- Segment Addition Postulate
- Angle Addition Postulate
- spherical geometry

You have created many conjectures in geometry. How can you reason more formally in Euclidean geometry with postulates and theorems?

GETTING STARTED

If . . ., Then . . .

Decide whether each statement is true or false and explain your reasoning.

1. All rectangles are quadrilaterals.

2. All rectangles are squares.

3. If it rains today, then it will not rain tomorrow.

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1.1 Counterexamples

In previous lessons—and in previous grades—you have used different kinds of reasoning to arrive at conclusions and to justify those conclusions. In this lesson and the next, you will start to use a more exact kind of reasoning commonly used in formal mathematics. When using this kind of reasoning, it is important to be able to identify false conclusions.

There are two reasons why a conclusion may be false. Either the assumed information is false, or the conclusion does not necessarily follow from the hypothesis.

Think about:

If you are reading this sentence, then your first language must be English. This is an invalid argument. Both the assumed information and conclusion might be true, but the conclusion does not necessarily follow from the given.

1. **Derek tells his little brother that it will not rain for the next 30 days because he "knows everything." Why is this conclusion false?**

2. **The sum of the interior angles of a figure is 360°, so the figure must be a square. Why is this conclusion false?**

3. **Write an example of a conclusion that is false because the assumed information is false.**

4. **Write an example of a conclusion that is false because the argument is not valid.**

To show that a statement is false, you can provide a *counterexample*. A **counterexample** is a specific example that shows that a general statement is not true.

5. Provide a counterexample for each statement to demonstrate that they are not true.

a. All prime numbers are odd.

b. The sum of the measures of two acute angles is always greater than 90º.

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1.2 Conditional Statements and Truth Tables

A conditional statement is a statement that can be written in the form "If p, then q." It can be written using symbols as $p \rightarrow q$, which is read as "p implies q." The hypothesis of a conditional statement is the variable p. A hypothesis is the "if" part of an "if-then" statement. The conclusion of a conditional statement is the variable q. A conclusion is the "then" part of an "if-then" statement.

The **truth value** of a conditional statement is whether the statement is true or false. If a conditional statement could be true, then the truth value of the statement is considered true. The truth value of a conditional statement is either true or false, but not both.

Worked Example

You can identify the hypothesis and conclusion from a conditional statement.

Conditional Statement

$$\underbrace{\text{If } x^2 = 36}_{\textbf{Hypothesis}},\ \underbrace{\text{then } x = 6 \text{ or } x = -6}_{\textbf{Conclusion}}.$$

1. Consider the conditional statement.
If the measure of an angle is 32°, then the angle is acute.

a. Identify the hypothesis *p*.

b. Identify the conclusion *q*.

Use the given conditional statement in Question 1 to answer Questions 2 through 5.

2. If *p* is true and *q* is true, then the truth value of a conditional statement is true.

a. What does the phrase "If *p* is true" mean in terms of the conditional statement?

b. What does the phrase "If *q* is true" mean in terms of the conditional statement?

c. Explain why the truth value of the conditional statement is true if both *p* and *q* are true.

3. **If *p* is true and *q* is false, then the truth value of a conditional statement is false.**

   a. **What does the phrase "If *p* is true" mean in terms of the conditional statement?**

Continue to use this conditional statement to respond to Questions 3 through 5.

If the measure of an angle is 32°, then the angle is acute.

   b. **What does the phrase "If *q* is false" mean in terms of the conditional statement?**

   c. **Explain why the truth value of the conditional statement is false if *p* is true and *q* is false.**

4. **If *p* is false and *q* is true, then the truth value of a conditional statement is true.**

   a. **What does the phrase "If *p* is false" mean in terms of the conditional statement?**

   b. **What does the phrase "If *q* is true" mean in terms of the conditional statement?**

Think about:

If *p* is false and *q* is true, the truth value is always true. Can you think of other examples that show this?

   c. **Explain why the truth value of the conditional statement is true if *p* is false and *q* is true.**

5. **If *p* is false and *q* is false, then the truth value of a conditional statement is true.**

   a. **What does the phrase "If *p* is false" mean in terms of the conditional statement?**

   b. **What does the phrase "If *q* is false" mean in terms of the conditional statement?**

   c. **Explain why the truth value of the conditional statement is true if both *p* and *q* are false.**

A **truth table** is a table that summarizes all possible truth values for a conditional statement $p \rightarrow q$. The first two columns of a truth table represent all possible truth values for the variables p and q. The last column represents the truth value of the conditional statement $p \rightarrow q$.

The truth values for the conditional statement "If the measure of an angle is 32°, then the angle is acute" are shown.

Worked Example

The truth value of the conditional statement $p \rightarrow q$ is determined by the truth value of p and the truth value of q.

- If p is true and q is true, then $p \rightarrow q$ is true.
- If p is true and q is false, then $p \rightarrow q$ is false.
- If p is false and q is true, then $p \rightarrow q$ is true.
- If p is false and q is false, then $p \rightarrow q$ is true.

p	q	$p \rightarrow q$
the measure of an angle is 32°	**the angle is acute**	**If the measure of an angle is 32°, then the angle is acute.**
T	T	T
T	F	F
F	T	T
F	F	T

6. **Consider the conditional statement.**
If $m\overline{AB} = 6$ inches and $m\overline{BC} = 6$ inches, then $\overline{AB} \cong \overline{BC}$.

a. **What is the hypothesis *p*?**

b. **What is the conclusion *q*?**

c. **If both *p* and *q* are true, what does that mean? What is the truth value of the conditional statement if both *p* and *q* are true?**

d. **If *p* is true and *q* is false, what does that mean? What is the truth value of the conditional statement if *p* is true and *q* is false?**

e. **If *p* is false and *q* is true, what does that mean? What is the truth value of the conditional statement if *p* is false and *q* is true?**

f. **If both *p* and *q* are false, what does that mean? What is the truth value of the conditional statement if both *p* and *q* are false?**

g. Summarize your answers to parts (a) through (f) by completing a truth table for the conditional statement.

p	q	$p \rightarrow q$

7. Mr. David wrote the statement shown on the board.

If $\overline{AC} \cong \overline{BC}$, then point C is the midpoint of $\overline{AB}$.

He asked his students to discuss the truth of this conditional statement.

Susan said she believed the statement to be true in all situations. Marcus disagreed with Susan and said that the statement was not true all of the time.

What is Marcus thinking and who is correct?

ACTIVITY
1.3

Postulates and Theorems

Greek mathematician Euclid is sometimes referred to as the Father of Geometry.

A postulate is a statement that is taken to be true without proof. A theorem is a statement that can be demonstrated to be true by accepted mathematical operations and arguments.

The Elements is a book written by the Greek mathematician Euclid. He used a small number of undefined terms and postulates to systematically prove many theorems. As a result, Euclid was able to develop a complete system we now know as **Euclidean geometry**.

Euclid's first five postulates are:

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn that has the segment as its radius and one endpoint as its center.
4. All right angles are congruent.
5. If two lines are drawn that intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. (This postulate is equivalent to what is known as the parallel postulate.)

Euclid used only the first four postulates to prove the first 28 propositions or theorems of *The Elements,* but was forced to use the fifth postulate, the parallel postulate, to prove the 29th theorem.

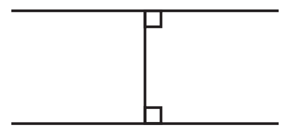
The Elements also includes five "common notions."

1. Things that equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things that coincide with one another equal one another.
5. The whole is greater than the part.

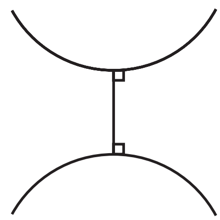
It is important to note that Euclidean geometry is not the only system of geometry Examples of non-Euclidian geometries include hyperbolic, elliptic and spherical geometry The essential difference between Euclidean and non-Euclidean geometry is the nature of parallel lines.

Another way to describe the differences between these geometries is to consider two lines in a plane that are both perpendicular to a third line.

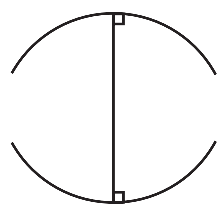
- In Euclidean geometry, the lines remain at a constant distance from each other and are known as parallels.



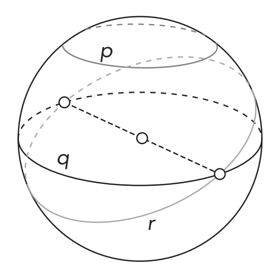
- In hyperbolic geometry, the lines "curve away" from each other.



- In elliptic geometry, the lines "curve toward" each other and eventually intersect.



Spherical geometry, like its name implies, is a geometry that substitutes a sphere for a plane, which makes it different from plane geometry in significant ways. In spherical geometry, some topics that you have learned about, such as parallel lines and the sum of the interior angles of a triangle, are very different.



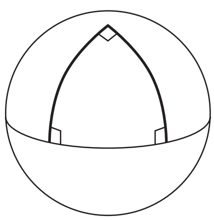
In spherical geometry, lines are defined as great circles of a sphere, which divide the sphere into two congruent hemispheres.

1. Which of the figures in the diagram shown—*p*, *q*, or *r*—appear to be spherical lines? Explain your reasoning.

2. Describe some similarities and differences between lines in Euclidean geometry and lines in spherical geometry, given this diagram.

3. **Do you think that parallel lines can exist in spherical geometry? Explain your reasoning.**

4. **At how many points do 2 different spherical lines intersect? Explain your reasoning.**



In spherical geometry, triangles have different properties as well. Consider the spherical triangle shown. It has 3 right angles, so the sum of its interior angle measures is 270°.

The sides of a spherical triangle are always arcs of great circles.

5. **Describe what you think are some similarities and differences between triangles in Euclidean geometry and spherical triangles, given this diagram.**

Investigate the properties of lines on a sphere and spherical triangles by drawing on a ball, such as a basketball or soccer ball.

6. **What do you think is the greatest sum of the interior angle measures of a spherical triangle? What is the least sum? Explain your reasoning.**

7. **What do you think would be the properties of a right spherical triangle? Explain your reasoning.**

You have already used the three undefined terms *point, line,* and *plane* to define related terms such as *line segment* and *angle.* Now consider these three fundamental postulates.

- The Linear Pair Postulate
- The Segment Addition Postulate
- The Angle Addition Postulate

You will use these postulates to make various conjectures. If you are able to prove your conjectures, then the conjectures will become theorems. These theorems can then be used to make even more conjectures, which may also become theorems. Mathematicians use this process to create new mathematical ideas.

The **Linear Pair Postulate** states: "If two angles form a linear pair, then the angles are supplementary."

8. Use the Linear Pair Postulate to complete each representation.

a. Sketch and label a linear pair.

b. Use your labeled sketch and the Linear Pair Postulate to write the hypothesis.

c. Use your labeled sketch and the Linear Pair Postulate to write the conclusion.

d. Use your conclusion and the definition of supplementary angles to write a statement about the measures of the angles in your figure.

The **Segment Addition Postulate** states: "If point *B* is on $\overline{AC}$ and between points *A* and *C*, then $AB + BC = AC$."

9. Use the Segment Addition Postulate to complete each representation.

a. Sketch and label collinear points *D*, *E*, and *F* with point *E* between points *D* and *F*.

b. Use your labeled sketch and the Segment Addition Postulate to write the hypothesis.

c. Use your labeled sketch and the Segment Addition Postulate to write the conclusion.

d. Write your conclusion using measure notation.

The **Angle Addition Postulate** states: "If point *D* lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$."

10. Use the Angle Addition Postulate to complete each representation.

a. Sketch and label $\angle DEF$ with $\overrightarrow{EG}$ drawn in the interior of $\angle DEF$.

b. Use your labeled sketch and the Angle Addition Postulate to write the hypothesis.

c. Use your labeled sketch and the Angle Addition Postulate to write the conclusion.

NOTES

TALK the TALK

While You Were Away . . .

1. **Write a short note to a friend explaining conditional statements, truth values, and truth tables. Include definitions of all terms and examples that are easy to understand.**