1

Elemental

Formal Reasoning in Euclidean Geometry

MATERIALS

None

Lesson Overview

Students are introduced to formal reasoning as a foundation for proving geometric theorems. They begin by writing a counterexample to demonstrate a statement is false. Students then analyze conditional statements, determine truth values for all possible cases, and summarize the results in a truth table. Euclidean geometry is introduced as a system built by postulates and proven theorems, and students analyze the Linear Pair Postulate, Segment Addition Postulate, and Angle Addition Postulate.

Geometry
Logical Argument and Constructions

(G.4) The student uses the process skills with deductive reasoning to understand geometric relationships. The student is expected to:

(A) distinguish between undefined terms, definitions, postulates, conjectures, and theorems.
(B) identify and determine the validity of the converse, inverse, and contrapositive of a conditional statement and recognize the connection between a biconditional statement and a true conditional statement with a true converse.
(C) verify that a conjecture is false using a counterexample.
(D) compare geometric relationships between Euclidean and spherical geometries, including parallel lines and the sum of the angles in a triangle.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 4.D, 4.J, 5.B, 5.F, 5.G

Essential Ideas

- The two reasons why a conclusion may be false is either the assumed information is false or the conclusion does not follow from the hypothesis.
- A counterexample is used to show a general statement is not true.
- A conditional statement is a statement that can be written in the form "If *p*, then *q*." The variable *p* represents the hypothesis and the variable *q* represents the conclusion.
- A truth value is whether or not a conditional statement is true or false; it is true if the conditional statement could be true, and it is false if the conditional statement could not be true.

- Truth tables are used to organize truth values of conditional statements.
- A postulate is a statement that is accepted without proof. A theorem is a statement that can be proven.
- The Linear Pair Postulate states: "If two angles form a linear pair, then the angles are supplementary."
- The Segment Addition Postulate states: "If point B is on $\overline{AC}$ and between points A and C, then $AB + BC = AC$."
- The Angle Addition Postulate states: "If point D lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$."
- There are multiple systems of geometry including Euclidean and non-Euclidean geometry.
- One example of non-Euclidean geometry is spherical geometry.

Lesson Structure and Pacing: 2 Days

Day 1

Engage

Getting Started: If . . ., Then . . .

Students decide whether statements are true or false and provide their reasoning. This activity is designed to engage students in thinking about the reasons why mathematical and other statements are true or false, valid or invalid.

Develop

Activity 1.1: Counterexamples

Students explore two reasons why a conclusion may be false: either the assumed information is false, or the argument is not valid. The term *counterexample* is defined, and students provide counterexamples to demonstrate that general statements are not true.

Activity 1.2: Conditional Statements and Truth Tables

Students are introduced to conditional statements, truth values, and truth tables. They analyze a conditional statement involving geometry to make sense of four possibilities and their truth values. Students then connect their process to a Worked Example that summarizes their responses in a truth table. They then repeat the process, including the creation of a truth table, with another geometry statement.

Day 2

Activity 1.3: Postulates and Theorems

Students are introduced to postulates, theorems, and how they are systematically built upon to develop Euclidean geometry. They compare Euclidean geometry to spherical geometry. Students analyze the Linear Pair Postulate, the Segment Addition Postulate, and the Angle Addition Postulate.

Demonstrate

Talk the Talk: While You Were Away . . .

Students write a note to a peer explaining conditional statements, truth values, and truth tables. They must create their own example within their explanation.

ENGAGE

Getting Started: If . . ., Then . . .

Facilitation Notes

In this activity, students decide whether statements are true or false and provide their reasoning. This activity is designed to engage students in thinking about the reasons why mathematical and other statements are true or false, valid or invalid.

Have the students work with a partner or in a group to complete Questions 1 through 3. Share responses as a class.

As students work, look for

Incorrect reasoning based upon a common property. If students reason that all rectangles are squares, they are focusing on a common property such as four congruent angles or the interior angles are right angles rather than the definition of rectangle and square.

Questions to ask

- What are the properties of a quadrilateral?
- What are the properties of a rectangle?
- Are all quadrilaterals also rectangles? Why or why not?
- What are the properties of a square?
- Are all squares also rectangles? Why or why not?
- Is the weather today and tomorrow always the same?
- Does the weather change day to day?
- How is the structure of the statements in Questions 1 and 2 different than the structure of the statement in Question 3?

Summary

Reasoning can be used to determine the accuracy of relationships based on definitions or if-then statements.

Activity 1.1
Counterexamples

DEVELOP

Facilitation Notes

In this activity, students explore two reasons why a conclusion may be false: either the assumed information is false, or the argument is not valid. The term *counterexample* is defined, and students provide counterexamples to demonstrate that general statements are not true.

Ask a student to read the introduction aloud. Discuss as a class.

Differentiation strategy

To assist all students, have them highlight and number the two reasons why a conclusion may be false.

Questions to ask

- What is the hypothesis of a statement?
- How do you identify the hypothesis of a statement?
- Does every statement have a hypothesis and a conclusion? Use the questions from the Getting Started to support your reasoning.
- What is a reason a conclusion may be false? What is another reason?
- What is meant by *the conclusion does not necessarily follow from the hypothesis*?

Have the students work with a partner or in a group to complete Questions 1 through 4. Share responses as a class.

Questions to ask

- Is this a situation when the assumed information is false, or is it a situation when the conclusion does not follow from the hypothesis?
- What is the assumed information in this situation? Why is it false?
- How does this lead to an incorrect conclusion?
- Why does the conclusion not follow from the hypothesis?
- What word or phrase could replace the term *must* so that the conclusion is true?
- What did you use for your false assumption?
- What is meant by *the argument is not valid*? Which of the two reasons why a conclusion is false have the same meaning?
- What invalid argument did you use?

Have the students work with a partner or in a group to read the definition after Question 4 and complete Question 5. Share responses as a class.

Differentiation strategy

To assist all students, discuss the meaning of the prefix *counter-*. Use the terms *clockwise* and *counterclockwise* to help students realize that the prefix *counter-* means opposite. Therefore, a counterexample is an example that proves the opposite, or disproves a statement.

Misconception

Students may assume that because one counterexample can be used to demonstrate that a statement is false, then several examples can be used to demonstrate that a statement is true. This is not the case. No number of examples is sufficient to demonstrate that a statement is true; instead, another type of reasoning is required.

Questions to ask

- What is the difference between an example and a counterexample?
- How many counterexamples are required to demonstrate that a statement is not true?
- How many examples do you think are required to demonstrate that a statement is true?
- What is the definition of a prime number?
- What is the one prime number that is even?
- What is another counterexample to demonstrate that the statement in Question 5, part (b) is not true?

Summary

A conclusion may be false because the assumed information is false or the conclusion does not necessarily follow from the hypothesis. A counterexample is a specific example that shows that a general statement is not true.

Activity 1.2

Conditional Statements and Truth Tables

Facilitation Notes

In this activity, students are introduced to conditional statements, truth values, and truth tables. They analyze a conditional statement involving geometry to make sense of four possibilities and their truth values. Students then connect their process to a Worked Example that summarizes their responses in a truth table. They then repeat the process, including the creation of a truth table, with another geometry statement.

Ask a student to read the definitions aloud. Discuss the definitions and Worked Example as a class.

Differentiation strategies

To assist all students,

- Set the stage for this activity by explaining the need for additional methods of reasoning. Counterexamples are useful to demonstrate that a statement is incorrect; however, formal methods, rather than examples, are required to demonstrate that a statement is correct.

- Help them make sense of the term *conditional* by relating it to its common use outside of math class. For example, "I will not assign homework tonight on the condition that we finish the entire lesson during class today." Ask students to rewrite the example as an *if p, then q* statement.
- Suggest they highlight the term *could* in the statement, "If a conditional statement *could* be true, then the truth value of the statement is considered true." Address the fact that the importance of this word may not be apparent immediately; however, its importance will become apparent as they respond to questions in this activity. Direct students to this statement as they respond to Questions 5 and 6 when *p* is false.
- While it is important that students understand and use the correct terminology regarding truth tables, it may be helpful as they are getting started to equate a truth value of *true* to mean *right*, and a truth value of *false* to mean *wrong*.

Misconceptions

- Students may not understand that the variables *p* and *q* may not represent numbers.
- Students may be confused by the fact that a truth value is not a number, but rather *true* or *false*.
- They may not understand that the truth value is related to the combination of the truth and/or falseness of both the *p* and *q* statements.

Questions to ask

- What does *p* represent?
- What does *q* represent?
- What does the direction of the arrow suggest?
- What variable is related to the hypothesis? The conclusion?
- If you had to give this conditional statement a truth value, what would it be? Why?
- How could you rewrite the conclusion so that the conditional statement has a truth value that is false?

Ask a student to read the conditional statement aloud. Complete Question 1 as a class.

Questions to ask

- How do you identify the hypothesis of a statement?
- How do you identify the conclusion of a statement?
- Use the framework of a conditional statement to explain why the words *if* and *then* are not included in the hypothesis or conclusion.

Have the students work with a partner or in a group to complete Questions 2 through 5. Share responses as a class.

Differentiation strategies

To assist all students,

- Prior to students beginning Questions 2 through 5, discuss the organization of the questions with the class.
    - For "If *p*, then *q*", the *p* component could be true or false, and the *q* component could be true or false. Ask students to list the four possibilities using the variables *p* and *q*, not the context of the given conditional statement.
    - Ask students to predict what they think the truth value of the entire conditional statement is based on the combination of true and false *p*'s and *q*'s. Display their predictions for later reference.
    - Then, redirect them to respond to the questions using the context of the conditional statement. As students go through the questions, have them highlight the introduction of each question and its truth value to more easily compare their predictions from the class discussion to the actual answers.
- Consider completing Question 2 as a class or having a class discussion once all students have completed Question 2 before they move on to the remaining questions.

Misconceptions

- Students may view the terms *acute* and *obtuse* as opposites, and think an equivalent way of expressing that *the angle is acute* is false is to say *the angle is obtuse*. The correct way to restate that *the angle is acute* is false is to say *the angle is not acute*.
- Students sometimes try to connect the results of a truth table with multiplication of integers; have them use the reasoning in Question 4 to clarify this misconception.

Questions to ask

- What is the definition of an acute angle?
- How can you use this definition to explain the conditional statement is true?
- Does it matter for *q is false* if you write *the angle is not acute* or *the angle is obtuse*? Explain why it makes a difference.
- If the angle is not acute, what other types of angle could it be?
- How can you use the definition of an acute angle to explain the conditional statement is false?
- How can you generalize the results from Questions 2 and 3, using the fact that *p* is true in both cases?
- Does it matter for *p is false* if you write *the angle is not 32*° or substitute a different degree measure? Explain why it makes a difference.

- Explain how the use of the term *could* in the explanation of a truth value at the start of this activity helps to make sense of *if p is false and q is true, then the truth value is true*.
- How can you generalize the results from Questions 4 and 5, using the fact that *p* is false in both cases?

Ask a student to read the definition aloud. Discuss the definition and Worked Example as a class.

Questions to ask

- Explain in your own words what a truth table is.
- What are the components of a truth table?
- What do the four rows with Ts and Fs represent?
- What do the entries F, F, T across a row represent?

Have the students work with a partner or in a group to complete Question 6. Share responses as a class.

Differentiation strategies

- To scaffold support, review the notation used. Discuss the difference between AB, $\overline{AB}$, and $m\overline{AB}$. Review the proper use of = and ≅.
- To extend the activity, ask pairs of students to create their own mathematical or non-mathematical conditional statement. Then, have students exchange their conditional statements and create a truth table for their classmates' conditional statements. Compare truth tables and discuss any issues.

Questions to ask

- Provide an example where *p is false and q is true* is a true statement.
- Are there times where *p is false and q is true* is a false statement? Provide an example.
- If *p is false and q is true* sometimes yields a true statement and sometimes yields a false statement, why is its truth value *true*?
- Provide an example where *p is false and q is false* is a true statement.
- Provide an example where *p is false and q is false* is a false statement.
- If *p is false and q is false* sometimes yields a true statement and sometimes yields a false statement, why is its truth value *true*?

Have the students work with a partner or in a group to complete Question 7. Share responses as a class.

Questions to ask

- What would a diagram of this conditional statement look like?
- If C is the midpoint of $\overline{AB}$, does that imply C is located on $\overline{AB}$?
- Just considering the hypothesis, draw another way the diagram could look.
- If $\overline{AC} \cong \overline{BC}$, does this mean point C is definitely located on $\overline{AB}$?

- What is another possible location for point C that is equidistant from points A and B?
- Why could point C be located anywhere on the perpendicular bisector of $\overline{AB}$?
- What is a possible diagram when $\overline{AC} \cong \overline{BC}$ is false?

Summary

A conditional statement is a statement that can be written in the form "If p, then q." A truth value of a conditional statement is whether the statement is true or false. A truth table is a table that summarizes all possible truth values for a conditional statement.

Activity 1.3
Postulates and Theorems

Facilitation Notes

In this activity, students are introduced to postulates, theorems, and how they are systematically built upon to develop Euclidean geometry. Students compare the difference between Euclidean and spherical geometry through lines and triangles. They analyze the Linear Pair Postulate, the Segment Addition Postulate, and the Angle Addition Postulate.

Ask students to read the definitions and information regarding Euclidean geometry through *The Elements*. Discuss as a class.

Differentiation strategies

To extend the lesson,

- Assign students the task of creating posters using diagrams to explain Euclid's first five postulates.
- Have students research non-Euclidean geometry.

Questions to ask

- What is the difference between a postulate and a theorem?
- How does your diagram make sense of the postulate?
- Explain the fifth postulate in your own words.
- Why do you think it makes sense that these postulates are accepted without proof?

Ask a student to read about non-Euclidean systems of geometry. Discuss as a class.

Have students read the information regarding spherical geometry. Discuss as a class.

Question to ask

- What is the key difference between Euclidean and non-Euclidean geometry?

Have students complete Questions 1 through 4. Share responses as a class.

Differentiation strategy

- To scaffold support, provide a globe or ball to help students visualize the difference between lines on a plane and lines on a sphere.

Misconception

- Students may think *p* is a spherical line because it is a circle. Remind students a spherical line must be a great circle which divides the sphere into two congruent hemispheres.

Questions to ask

- Would the shortest distance between two points be an arc in spherical geometry?
- Are line infinite in spherical geometry?

Have students work with a partner or in a group to complete Questions 5 through 7. Share responses as a class.

Misconception

Students have previously been taught that the sum of the interior angles of a triangle is 180°. Make sure they understand that in spherical geometry, the sum of the interior angles of a triangle can vary.

Questions to ask

- What other geometric figures can you think of that would have different properties in spherical geometry?
- What properties do you think they would have?

Ask a student to read the introductory information about the three undefined terms and three fundamental postulates.

Have students work with a partner or in a group to complete Questions 8 through 10. Share responses as a class.

Differentiation strategy

As an alternative grouping method, assign each group of students one of the three postulates in Questions 8 through 10. Have each group become an expert on their postulate by answering the questions, creating a poster, teaching the class their postulate, and developing questions to ask the class about their postulate. After each presentation, allow the class a few minutes to complete the accompanying questions in the textbook.

For Question 8:

As students work, look for

- Appropriate use of math terms, such as *adjacent*, *vertex*, and *rays*.
- Linear pairs drawn in different orientations.

Misconception

Students may think that linear pairs must be drawn as two rays in opposite horizontal directions. If so, clarify the misconception by drawing a letter X and asking students to identify the different linear pairs formed.

Questions to ask

- What is a linear pair of angles?
- Does a linear pair of angles always share a common ray? A common vertex?
- What is the difference between supplementary and complementary angles?
- Can two angles be supplementary, but not form a linear pair?
- Can two angles form a linear pair, but not be supplementary?
- How do labels make it easier to discuss mathematical diagrams?

For Question 9:

As students work, look for

- Confusion about why the letters D, E, and F are used rather than A, B, and C. The postulate is a general statement to describe a situation.
- Different locations of E on $\overline{DF}$.
- Questions as to what is meant by *measure notation*. The expectation is that students write their answers using m for measure, segment symbols, and equal signs.

Misconception

Students may think that *between* implies the use of a midpoint. Address this error in thinking by making a comparison using numbers. Ask "I am thinking of a number between 1 and 5, what is my number?" Discuss that the answer is not necessarily the middle number, 3.

Questions to ask

- Why is it important to state that point E must be on $\overline{DF}$ between points D and F?
- Does point E have to be the midpoint of $\overline{DF}$? Explain why not.
- Why can't we just say "If points D, E, and F are collinear, then $DE + EF = DF$?"
- How does this geometry postulate relate to solving algebraic equations?

For Question 10:

As students work, look for

- E as the vertex of the angle.
- Different locations for $\overrightarrow{EG}$.

Misconception

Students may not realize that $\overrightarrow{EG}$ does not need to be the angle bisector. If so, explain how *in the interior* for angles is analogous to *between* for line segments.

Questions to ask

- Why is it important to state that point *G* must be on the interior of $\angle DEF$?
- Does point *G* have to lie on the angle bisector of $\angle DEF$? Explain why not.
- How are the Segment Addition Postulate and Angle Addition Postulate related?
- How can this postulate be applied to subtracting angles?

Summary

A postulate is a statement that is taken to be true without proof. A theorem is a statement that can be demonstrated to be true by accepted mathematical operations and arguments. Euclidean geometry is built upon postulates and proven theorems.

Talk the Talk: While You Were Away . . .

DEMONSTRATE

Facilitation Notes

In this activity, students write a note to a peer explaining conditional statements, truth values, and truth tables. They must create their own example within their explanation.

Have the students work with a partner or in a group to complete this activity. Share responses as a class.

Differentiation strategy

To assist all students, stress generalizing truth values based upon whether the hypothesis, *p*, is true or false and then whether the concluding statement, *q*, is true or false. This way, students can use the reasoning that if a conditional statement could be true, then the truth value is true. If students try to start with the truth value, and then determine the combinations of *p* and *q* that yield these values, they often resort to memorization rather than meaning.

As students work, look for

- Errors in conditional statements. Incorrect conclusions may be due to assumed information that is false or a conclusion that does not necessarily follow from the hypothesis.
- Math examples and non-math examples.
- Clear explanations using examples to explain truth values.
- Generalizing with accurate reasoning.

Misconception

Students may assume the hypothesis must always be written first followed by the conclusion. Show students how their statement could be rewritten stating the conclusion first.

Questions to ask

- Do all conditional statements have a hypothesis and conclusion?
- How can you distinguish the hypothesis from the conclusion?
- If the hypothesis is true, how can you determine the truth value of the conditional statement?
- If the hypothesis is false, how can you determine the truth value of the conditional statement?
- Why is the truth value of a conditional statement always true when the hypothesis is false?
- Are all truth tables similar? How so? Why does this happen?

Summary

A conditional statement is a statement that can be written in the form "If p, then q." The truth value of a conditional statement is whether the statement is true or false. If p is true, then q should also be true for the conditional statement to be true; if q is false, then the conditional statement is false. If p is false, the conditional statement is true regardless of whether q is false or not.

1

Elemental

Formal Reasoning in Euclidean Geometry

Warm Up

Solve for x.

1. $x + 4x = 90$

2. $0.5x + 2x = 90$

3. $x + 2x = 180$

4. $0.5x + 4x = 180$

Learning Goals

- Identify the hypothesis and conclusion of a conditional statement.
- Explore the truth values of conditional statements.
- Use a truth table.
- Differentiate between postulates and theorems.
- Differentiate between Euclidean and spherical geometry.

Key Terms

- counterexample
- truth value
- truth table
- Euclidean geometry
- Linear Pair Postulate
- Segment Addition Postulate
- Angle Addition Postulate
- spherical geometry

You have created many conjectures in geometry. How can you reason more formally in Euclidean geometry with postulates and theorems?

Warm Up Answers

1. $x = 18$
2. $x = 36$
3. $x = 60$
4. $x = 40$

Answers

1. True
2. False
3. False

GETTING STARTED

If . . ., Then . . .

Decide whether each statement is true or false and explain your reasoning.

1. **All rectangles are quadrilaterals.**

2. **All rectangles are squares.**

3. **If it rains today, then it will not rain tomorrow.**

2 • TOPIC 3: Congruence Through Transformations

ACTIVITY
1.1

Counterexamples

In previous lessons—and in previous grades—you have used different kinds of reasoning to arrive at conclusions and to justify those conclusions. In this lesson and the next, you will start to use a more exact kind of reasoning commonly used in formal mathematics. When using this kind of reasoning, it is important to be able to identify false conclusions.

There are two reasons why a conclusion may be false. Either the assumed information is false, or the conclusion does not necessarily follow from the hypothesis.

1. Derek tells his little brother that it will not rain for the next 30 days because he "knows everything." Why is this conclusion false?

2. The sum of the interior angles of a figure is 360°, so the figure must be a square. Why is this conclusion false?

3. Write an example of a conclusion that is false because the assumed information is false.

4. Write an example of a conclusion that is false because the argument is not valid.

Think about:

If you are reading this sentence, then your first language must be English. This is an invalid argument. Both the assumed information and conclusion might be true, but the conclusion does not necessarily follow from the given.

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Answers

1. The conclusion is false because the assumed information is false. No one knows everything.
2. The conclusion is false because the argument is not valid. The figure doesn't have to be a square, it could be any type of quadrilateral.
3. Sample answer. All kids like broccoli. Max is a kid. So, Max likes broccoli.
4. Sample answer. Kayleigh doesn't like green fruit. An apple is a green fruit. So, Kayleigh doesn't like apples.

ELL Tip

Assess students' prior knowledge of the term *invalid*. First discuss the definition of the base word *valid*, which means *to have a sound basis in logic or fact*. Remind students that while the prefix *in-* has multiple meanings, it can be used to negate a word, or show the opposite of a word. Read aloud the first two sentences in the *Think About* section of the activity, "If you are reading this sentence, then your first language must be English. This is an *invalid* argument." Discuss the context of *invalid* in the second sentence, as *an argument that does not have sound logic or facts to back it up*.

Answers

5a. The only counterexample is 2, which is a prime number that is even.

5b. Sample answer. Two angles that each measure 35° are acute angles, but the sum of their measures is 70°, which is less than 90°.

To show that a statement is false, you can provide a *counterexample.* A **counterexample** is a specific example that shows that a general statement is not true.

5. Provide a counterexample for each statement to demonstrate that they are not true.

a. All prime numbers are odd.

b. The sum of the measures of two acute angles is always greater than 90°.

ACTIVITY 1.2 Conditional Statements and Truth Tables

A conditional statement is a statement that can be written in the form "If p, then q." It can be written using symbols as $p \rightarrow q$, which is read as "p implies q." The hypothesis of a conditional statement is the variable p. A hypothesis is the "if" part of an "if-then" statement. The conclusion of a conditional statement is the variable q. A conclusion is the "then" part of an "if-then" statement.

The **truth value** of a conditional statement is whether the statement is true or false. If a conditional statement could be true, then the truth value of the statement is considered true. The truth value of a conditional statement is either true or false, but not both.

Worked Example

You can identify the hypothesis and conclusion from a conditional statement.

Conditional Statement

$\underbrace{\text{If } x^2 = 36}_{\textbf{Hypothesis}}$, $\underbrace{\text{then } x = 6 \text{ or } x = -6}_{\textbf{Conclusion}}$.

1. Consider the conditional statement.
If the measure of an angle is 32°, then the angle is acute.

a. Identify the hypothesis *p*.

b. Identify the conclusion *q*.

Use the given conditional statement in Question 1 to answer Questions 2 through 5.

2. If *p* is true and *q* is true, then the truth value of a conditional statement is true.

a. What does the phrase "If *p* is true" mean in terms of the conditional statement?

b. What does the phrase "If *q* is true" mean in terms of the conditional statement?

c. Explain why the truth value of the conditional statement is true if both *p* and *q* are true.

Answers

1a. The measure of an angle is 32°.

1b. The angle is acute.

2a. "If *p* is true" means that the measure of the angle is 32°.

2b. "If *q* is true" means that the angle is acute.

2c. The truth value of the conditional statement is true because, by definition, an acute angle is an angle whose measure is less than 90°. An angle whose measure is 32° has a measure less than 90° and must be an acute angle.

Answers

3a. "If p is true" means that the measure of the angle is 32°.

3b. "If q is false" means that the angle is not acute.

3c. The truth value of the conditional statement is false because, by definition, an acute angle is an angle whose measure is less than 90°. The angle is 32°, so the statement that the angle is not acute is a false statement.

4a. "If p is false" means that the measure of the angle is not 32°.

4b. "If q is true" means that the angle is acute.

4c. The truth value of the conditional statement is true because, by definition, an acute angle is an angle whose measure is less than 90°. The angle could be less than 90°, so the statement that the angle is acute is a true statement.

Continue to use this conditional statement to respond to Questions 3 through 5.

If the measure of an angle is 32°, then the angle is acute.

3. If *p* is true and *q* is false, then the truth value of a conditional statement is false.

a. What does the phrase "If *p* is true" mean in terms of the conditional statement?

b. What does the phrase "If *q* is false" mean in terms of the conditional statement?

c. Explain why the truth value of the conditional statement is false if *p* is true and *q* is false.

4. If *p* is false and *q* is true, then the truth value of a conditional statement is true.

a. What does the phrase "If *p* is false" mean in terms of the conditional statement?

b. What does the phrase "If *q* is true" mean in terms of the conditional statement?

c. Explain why the truth value of the conditional statement is true if *p* is false and *q* is true.

Think about:

If p is false and q is true, the truth value is always true. Can you think of other examples that show this?

5. **If p is false and q is false, then the truth value of a conditional statement is true.**

   a. **What does the phrase "If p is false" mean in terms of the conditional statement?**

   b. **What does the phrase "If q is false" mean in terms of the conditional statement?**

   c. **Explain why the truth value of the conditional statement is true if both p and q are false.**

A **truth table** is a table that summarizes all possible truth values for a conditional statement $p \rightarrow q$. The first two columns of a truth table represent all possible truth values for the variables p and q. The last column represents the truth value of the conditional statement $p \rightarrow q$.

The truth values for the conditional statement "If the measure of an angle is 32°, then the angle is acute" are shown.

Worked Example

The truth value of the conditional statement $p \rightarrow q$ is determined by the truth value of p and the truth value of q.

- If p is true and q is true, then $p \rightarrow q$ is true.
- If p is true and q is false, then $p \rightarrow q$ is false.
- If p is false and q is true, then $p \rightarrow q$ is true.
- If p is false and q is false, then $p \rightarrow q$ is true.

p	q	$p \rightarrow q$
the measure of an angle is 32°	**the angle is acute**	**If the measure of an angle is 32°, then the angle is acute.**
T	T	T
T	F	F
F	T	T
F	F	T

Answers

5a. "If p is false" means that the measure of the angle is not 32°.

5b. "If q is false" means that the angle is not acute.

5c. The truth value of the conditional statement is true because, by definition, an acute angle is an angle whose measure is less than 90°. The angle could be greater than 90°, so the statement that the angle is not acute is a true statement.

Answers

6a. $m\overline{AB} = 6$ inches and $m\overline{BC} = 6$ inches.

6b. $\overline{AB} \cong \overline{BC}$

6c. If both p and q are true, this means $m\overline{AB} = 6$ inches and $m\overline{BC} = 6$ inches; therefore, $\overline{AB} \cong \overline{BC}$. The truth value of the conditional statement is true.

6d. If p is true and q is false, this means $m\overline{AB} = 6$ inches and $m\overline{BC} = 6$ inches; therefore, $\overline{AB} \ncong \overline{BC}$. The truth value of the conditional statement is false.

6e. If p is false and q is true, this means $m\overline{AB} \neq 6$ inches or $m\overline{BC} \neq 6$ inches; therefore, $\overline{AB} \cong \overline{BC}$. The truth value of the conditional statement is true. This statement could be true if both $m\overline{AB}$ and $m\overline{BC}$ equal the same value, but not 6 inches.

6f. If both p and q are false, $m\overline{AB} \neq 6$ inches or $m\overline{BC} \neq 6$ inches; therefore, $\overline{AB} \ncong \overline{BC}$. The truth value of the conditional statement is true. This statement could be true if $m\overline{AB}$ and $m\overline{BC}$ do not equal the same value.

6. Consider the conditional statement.
If $m\overline{AB} = 6$ inches and $m\overline{BC} = 6$ inches, then $\overline{AB} \cong \overline{BC}$.

a. What is the hypothesis *p*?

b. What is the conclusion *q*?

c. If both *p* and *q* are true, what does that mean? What is the truth value of the conditional statement if both *p* and *q* are true?

d. If *p* is true and *q* is false, what does that mean? What is the truth value of the conditional statement if *p* is true and *q* is false?

e. If *p* is false and *q* is true, what does that mean? What is the truth value of the conditional statement if *p* is false and *q* is true?

f. If both *p* and *q* are false, what does that mean? What is the truth value of the conditional statement if both *p* and *q* are false?

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g. **Summarize your answers to parts (a) through (f) by completing a truth table for the conditional statement.**

p	q	$p \rightarrow q$

7. **Mr. David wrote the statement shown on the board.**

If $\overline{AC} \cong \overline{BC}$, then point C is the midpoint of $\overline{AB}$.

He asked his students to discuss the truth of this conditional statement.

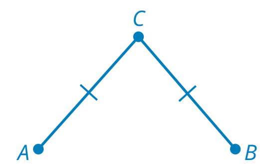
Susan said she believed the statement to be true in all situations. Marcus disagreed with Susan and said that the statement was not true all of the time.

What is Marcus thinking and who is correct?

Answers

6g. See table below.

7. Marcus is correct. If point C is not located on $\overline{AB}$, the hypothesis may be true but the conclusion is false as shown. Point C can lie anywhere on the perpendicular bisector of $\overline{AB}$.



6g.

p	q	$p \rightarrow q$
$m\overline{AB} = 6$ inches and $m\overline{BC} = 6$ inches	$\overline{AB} \cong \overline{BC}$	If $m\overline{AB} = 6$ inches and $m\overline{BC} = 6$ inches, then $\overline{AB} \cong \overline{BC}$.
T	T	T
T	F	F
F	T	T
F	F	T

ACTIVITY 1.3

Postulates and Theorems

Greek mathematician Euclid is sometimes referred to as the Father of Geometry.

A postulate is a statement that is taken to be true without proof. A theorem is a statement that can be demonstrated to be true by accepted mathematical operations and arguments.

The Elements is a book written by the Greek mathematician Euclid. He used a small number of undefined terms and postulates to systematically prove many theorems. As a result, Euclid was able to develop a complete system we now know as **Euclidean geometry**.

Euclid's first five postulates are:

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn that has the segment as its radius and one endpoint as its center.
4. All right angles are congruent.
5. If two lines are drawn that intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. (This postulate is equivalent to what is known as the parallel postulate.)

Euclid used only the first four postulates to prove the first 28 propositions or theorems of *The Elements,* but was forced to use the fifth postulate, the parallel postulate, to prove the 29th theorem.

The Elements also includes five "common notions."

1. Things that equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things that coincide with one another equal one another.
5. The whole is greater than the part.

It is important to note that Euclidean geometry is not the only system of geometry Examples of non-Euclidian geometries include hyperbolic, elliptic and spherical geometry The essential difference between Euclidean and non-Euclidean geometry is the nature of parallel lines.

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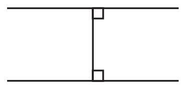
ELL Tip

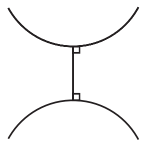
Determine whether students are familiar with the term *systematically*. If not, review the definition of a *system*, which is *a set of parts which interact to create a whole*. Read aloud the fourth sentence in the activity, "He used a small number of undefined terms and postulates to *systematically* prove many theorems." Encourage a discussion about the application of *systematically* in this sentence, and ensure students' understanding that the theorems were proven using a combination of parts (undefined terms and postulates).

Another way to describe the differences between these geometries is to consider two lines in a plane that are both perpendicular to a third line.

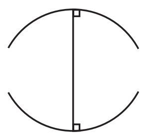
- In Euclidean geometry, the lines remain at a constant distance from each other and are known as parallels.



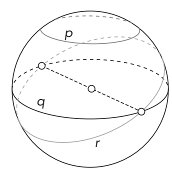
- In hyperbolic geometry, the lines "curve away" from each other.



- In elliptic geometry, the lines "curve toward" each other and eventually intersect.



Spherical geometry, like its name implies, is a geometry that substitutes a sphere for a plane, which makes it different from plane geometry in significant ways. In spherical geometry, some topics that you have learned about, such as parallel lines and the sum of the interior angles of a triangle, are very different.



In spherical geometry, lines are defined as great circles of a sphere, which divide the sphere into two congruent hemispheres.

1. Which of the figures in the diagram shown—*p*, *q*, or *r*—appear to be spherical lines? Explain your reasoning.

2. Describe some similarities and differences between lines in Euclidean geometry and lines in spherical geometry, given this diagram.

Answers

1. Figures q and r are spherical lines because they are great circles. Figure p is not a line.
2. In Euclidean geometry, lines are straight and extend infinitely in two directions. In spherical geometry, lines are great circles of a sphere. These lines appear to extend infinitely in two directions also, but a line in spherical geometry overlaps itself.

ELL Tip

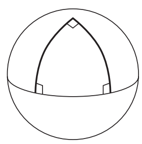
To better understand spherical geometry, have students relate to familiar objects like a globe or ball to help students visualize the difference between the lines on a plane and lines on a sphere.

Answers

3. Parallel lines do not exist, because lines are defined as great circles. Any two different great circles of a sphere must intersect.
4. Because spherical lines are great circles, two different spherical lines intersect at 2 points.
5. Answers will vary. The sum of the interior angle measures of a triangle in Euclidean geometry is 180°. Spherical triangles do not necessarily have interior angle measures whose sum is 180°. The sides of a Euclidean triangle are all straight line segments, and the sides of a spherical triangle are curved line segments.
6. Answers will vary. The sum of the interior angle measures of a spherical triangle is greater than 180° and less than 540°.
7. A right spherical triangle could be a triangle with just one 90-degree angle.

3. Do you think that parallel lines can exist in spherical geometry? Explain your reasoning.

4. At how many points do 2 different spherical lines intersect? Explain your reasoning.



In spherical geometry, triangles have different properties as well. Consider the spherical triangle shown. It has 3 right angles, so the sum of its interior angle measures is 270°.

The sides of a spherical triangle are always arcs of great circles.

5. Describe what you think are some similarities and differences between triangles in Euclidean geometry and spherical triangles, given this diagram.

Investigate the properties of lines on a sphere and spherical triangles by drawing on a ball, such as a basketball or soccer ball.

6. What do you think is the greatest sum of the interior angle measures of a spherical triangle? What is the least sum? Explain your reasoning.

7. What do you think would be the properties of a right spherical triangle? Explain your reasoning.

You have already used the three undefined terms *point, line,* and *plane* to define related terms such as *line segment* and *angle.* Now consider these three fundamental postulates.

- The Linear Pair Postulate
- The Segment Addition Postulate
- The Angle Addition Postulate

You will use these postulates to make various conjectures. If you are able to prove your conjectures, then the conjectures will become theorems. These theorems can then be used to make even more conjectures, which may also become theorems. Mathematicians use this process to create new mathematical ideas.

The **Linear Pair Postulate** states: "If two angles form a linear pair, then the angles are supplementary."

8. Use the Linear Pair Postulate to complete each representation.

a. Sketch and label a linear pair.

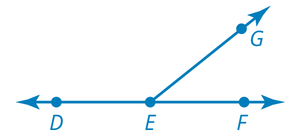
b. Use your labeled sketch and the Linear Pair Postulate to write the hypothesis.

c. Use your labeled sketch and the Linear Pair Postulate to write the conclusion.

d. Use your conclusion and the definition of supplementary angles to write a statement about the measures of the angles in your figure.

Answers

8a. Sample answer.



8b. Angles *DEG* and *GEF* form a linear pair.

8c. Angles *DEG* and *GEF* are supplementary angles.

8d. $m\angle DEG + m\angle GEF = 180°$

ELL Tip

Assess students' prior knowledge of the term *coincide*. Remind them that the prefix *co-* means *with* or *together*. Define *coincide* as *to happen at or during the same time*. Provide examples of the application of *coincide*, such as *two birthday parties coinciding, the arrival of two airplanes coinciding*, and *the after school activities of three children coinciding*. Discuss the fourth "common notion" of *The Elements*, "Things that *coincide* with one another equal one another." Clarify any remaining misunderstandings students may have about the context of *coincide* in the sentence.

Answers

9a. Sample answer.

9b. Point E is on $\overline{DF}$ and between points D and F.

9c. $DE + EF = DF$

9d. $m\overline{DE} + m\overline{EF} = m\overline{DF}$

The **Segment Addition Postulate** states: "If point B is on $\overline{AC}$ and between points A and C, then $AB + BC = AC$."

9. Use the Segment Addition Postulate to complete each representation.

a. Sketch and label collinear points *D*, *E*, and *F* with point *E* between points *D* and *F*.

b. Use your labeled sketch and the Segment Addition Postulate to write the hypothesis.

c. Use your labeled sketch and the Segment Addition Postulate to write the conclusion.

d. Write your conclusion using measure notation.

The **Angle Addition Postulate** states: "If point D lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$."

10. Use the Angle Addition Postulate to complete each representation.

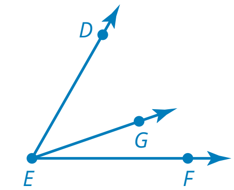
a. Sketch and label $\angle DEF$ with $\overrightarrow{EG}$ drawn in the interior of $\angle DEF$.

b. Use your labeled sketch and the Angle Addition Postulate to write the hypothesis.

c. Use your labeled sketch and the Angle Addition Postulate to write the conclusion.

Answers

10a. Sample answer.



10b. Point G lies in the interior of $\angle DEF$.

10c. $m\angle DEG + m\angle GEF = m\angle DEF$

Answers

1. Sample answer.
   A conditional statement is an "If *p*, then *q*" statement. The hypothesis is *p*, and the conclusion is *q*. For example, if the two numbers are 2 and 6, then the product is 12.
   The truth value of a conditional statement is whether the statement is true or false, based upon whether *p* is true or false and *q* is true or false. There are four combinations.
   If *p* is true, and *q* is true, then the conditional statement is true.
   If the two numbers are 2 and 6, then *the product is 12* is a true statement.
   If *p* is true, and *q* is false, then the conditional statement is false.
   If the two numbers are 2 and 6, then *the product is not 12* is a false statement.
   If *p* is false, and *q* is true, then the conditional statement is true.
   If the two numbers are not 2 and 6, then *the product is 12* is a true statement.
   This is true because the two numbers could be other values that have a product of 12, such as 3 and 4.
   If *p* is false, and *q* is false, then the conditional statement is true.
   If the two numbers are not 2 and 6, then *the product is not 12* is a true statement.

NOTES

TALK the TALK

While You Were Away . . .

1. **Write a short note to a friend explaining conditional statements, truth values, and truth tables. Include definitions of all terms and examples that are easy to understand.**

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This is true because the two numbers could be other values that do not have a product of 12, such as 3 and 5 that have a product of 15.
A truth table is a table that summarizes all possible truth values for a conditional statement.