2

ASA, SAS, and SSS

Proving Triangle Congruence Theorems

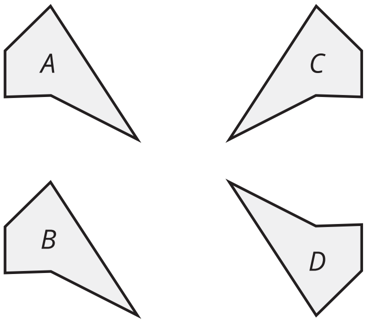
Warm Up

Describe transformations that map Figure *A* onto the other figures shown.

1. Figure *B*

2. Figure *C*

3. Figure *D*



Learning Goals

- Use the definition of congruence in terms of rigid motions to show that two triangles are congruent.
- Prove the Side-Side-Side Congruence Theorem using rigid motion transformations.
- Prove the Side-Angle-Side Congruence Theorem using rigid motion transformations.
- Prove the Angle-Side-Angle Congruence Theorem using rigid motion transformations.

Key Terms

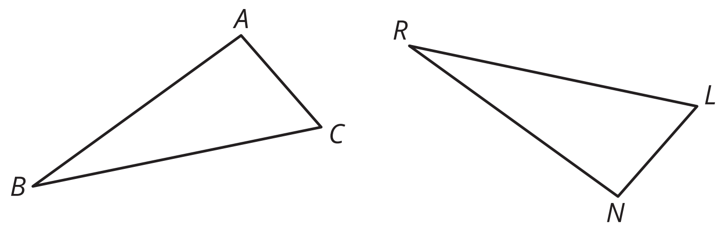
- Side-Side-Side Congruence Theorem (SSS)
- corresponding parts of congruent triangles are congruent (CPCTC)
- Side-Angle-Side Congruence Theorem (SAS)
- included angle
- Angle-Side-Angle Congruence Theorem (ASA)
- included side

You have defined the transformations that produce isometries. How can you use isometries to prove congruence theorems?

GETTING STARTED

Necessary Conditions

Consider the two triangles shown.



Each triangle pair has 6 relationships—3 pairs of sides and 3 pairs of angles. If the two triangles are congruent, all the corresponding side lengths and all the corresponding angle measures must be equal.

1. Use a ruler and protractor to determine whether the two triangles are congruent. Explain your strategy.

2. What is the minimum number of measurements you could make to determine whether the two triangles are congruent? Explain your reasoning.

ACTIVITY
2.1

Congruent Line Segments by Reflection

Congruent line segments and congruent angles are often denoted using special markers, rather than given measurements.

Slash markers can be used to indicate congruent line segments. When multiple line segments contain a single slash marker, this implies that all of those line segments are congruent. Double and triple slash markers can also be used to denote other line segment congruencies.

Arc markers can be used to indicate congruent angles. When multiple angles contain a single arc marker, this implies that those angles are congruent. Double and triple arc markers can also be used to denote other angle congruencies.

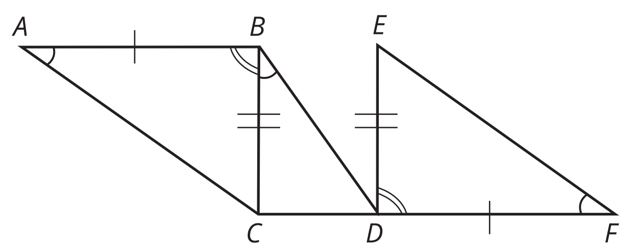
Worked Example

The markers on the diagram indicate congruent line segments.

$\overline{AB} \cong \overline{FD}$ and $\overline{BC} \cong \overline{DE}$

$\angle BAC \cong \angle CBD \cong \angle DFE$

$\angle ABC \cong \angle FDE$

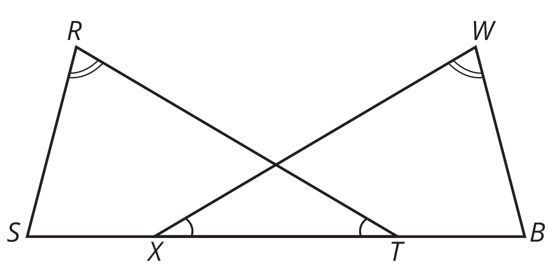


Think about:

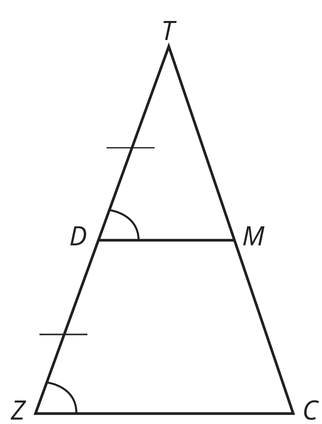
Although $\overline{DF}$ and $\overline{FD}$ represent the same line segment, when writing a congruence statement for segments that are part of a larger figure, think about how the sides and angles in the figure correspond to one another.

1. Write the congruence statements represented by the markers in each diagram.

a.



b.



Think about:

Make sure you are properly naming angles.

Remember:

A proof is a series of statements and corresponding reasons forming a valid argument that starts with a hypothesis and arrives at a conclusion.

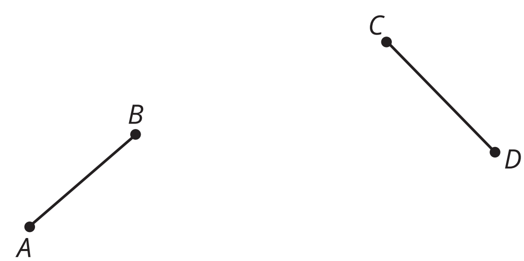
In previous lessons, you learned that:

(1) Isometries preserve distances and angle measures.
(2) Any point in the plane can be reflected across a line to map onto another point in the plane.
(3) A point is equidistant from two other points if and only if it lies on their perpendicular bisector.

You can use these facts to prove a conjecture that you have explored. Prove that you can always map one congruent line segment onto the other using at most two reflections.

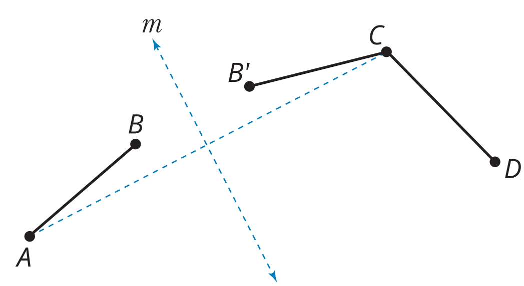
Worked Example

Prove that a segment can be mapped onto itself in at most two reflections.



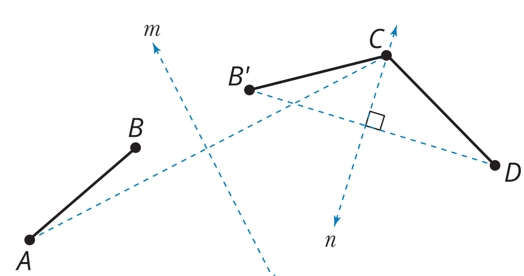
Suppose that $\overline{AB} \cong \overline{CD}$.

Since any point in the plane can be reflected across a line to map onto another point in the plane, you know that point C is a reflection of point A across line m. Reflect $\overline{AB}$ across line m, $R_m(\overline{AB}) \rightarrow (\overline{CB'})$.



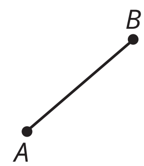
Reflections preserve distances, so you know that $\overline{CB'} \cong \overline{CD}$, because $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{CB'}$. Thus, point C is equidistant from point B' and from point D.

Since a point is equidistant from two other points if and only if it lies on their perpendicular bisector, this means that point C lies on the perpendicular bisector of $\overline{B'D}$ (line n).



Thus, one last reflection across the perpendicular bisector maps $\overline{CB'}$ onto $\overline{CD}$, $R_n(\overline{CB'}) \rightarrow (\overline{CD})$.

2. The proof in the Worked Example shows two reflections. Create an example in which $\overline{AB}$ maps onto $\overline{CD}$ in just one reflection. Explain your example.



3. Use the Worked Example to explain why you do not need more than two reflections to map a line segment onto a congruent line segment in the plane.

ACTIVITY

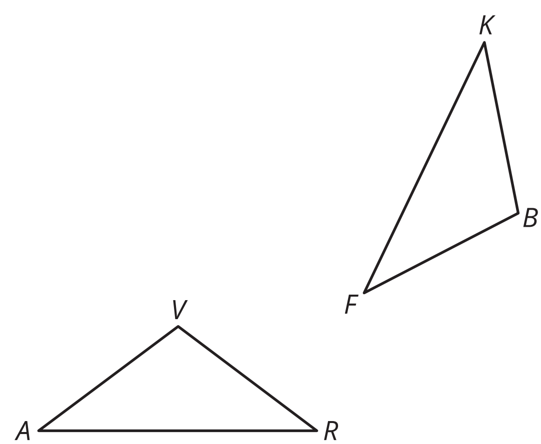
2.2 Side-Side-Side Congruence

In previous courses, you investigated the conditions necessary for forming a triangle with different side lengths. Now you will prove that triangles are congruent given different minimal criteria.

> You can use the fact that a segment can be mapped onto itself in at most two reflections to prove $\triangle VAR$ is congruent to $\triangle BKF$.

Consider two triangles such that the three sides of one triangle are congruent to the three sides of the second triangle. You can prove that this criteria is sufficient to demonstrate the two triangles are congruent.

Given $\triangle VAR$ and $\triangle BKF$. Suppose $VA = BK$, $VR = BF$, and $AR = KF$.



1. Complete the steps to show the proof that $\triangle VAR \cong \triangle BKF$.

a. Draw the reflection of $\triangle VAR$ across a line that maps point V onto point B. Label the image as $\triangle V'A'R'$. Give the reason(s) you can create this reflection.

b. Draw the reflection of $\triangle V'A'R'$ which maps $\overline{V'A'}$ onto $\overline{BK}$. Label the image as $\triangle V''A''R''$. Give the reason(s) you can create this reflection.

2. **Summarize the proof you completed in Question 1. Explain how knowing that the three corresponding sides of two triangles are congruent proves that the two triangles are congruent.**

Because this relationship has been proved to be true, you may now refer to it as a theorem. The **Side-Side-Side Congruence Theorem (SSS)** states: "If three sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent."

Any theorem can be used in the future as a reason in other proofs.

If two triangles are congruent, then each part of one triangle is congruent to the corresponding part of the other triangle. **Corresponding parts of congruent triangles are congruent**, abbreviated as **CPCTC**, is often used as a reason in proofs. CPCTC states that corresponding angles and sides in two congruent triangles are congruent. This reason can only be used after you have proven that the triangles are congruent.

3. **Write congruence statements for all corresponding side and angle relationships of the pre-image and the image.**

ACTIVITY 2.3

Side-Angle-Side Congruence

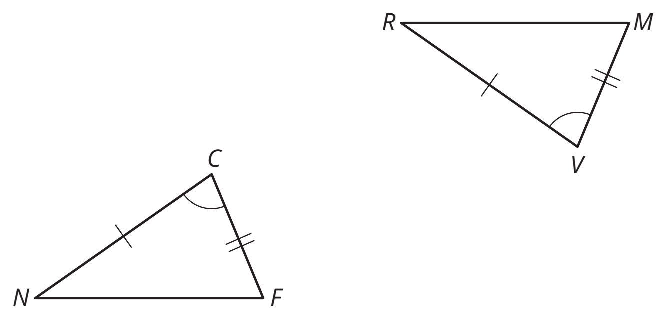
An **included angle** is the angle formed by two sides of a triangle.

Consider two triangles that have two sides and an included angle congruent. Analyze the proof using rigid motion transformations to demonstrate that this criteria is sufficient to prove the two triangles are congruent.

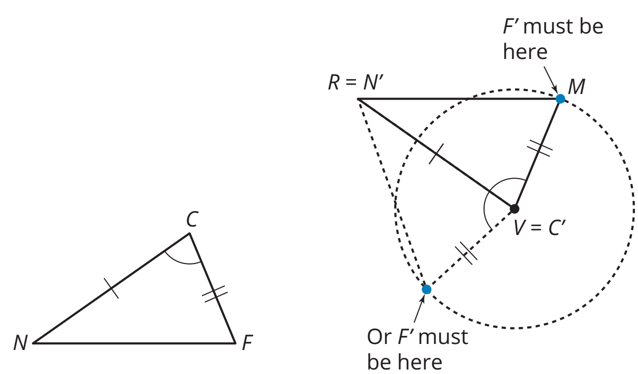
Worked Example

Given $\triangle CNF$ and $\triangle VRM$, $\overline{CN} \cong \overline{VR}$, $\overline{CF} \cong \overline{VM}$, and $\angle C \cong \angle V$. Prove $\triangle CNF \cong \triangle VRM$.

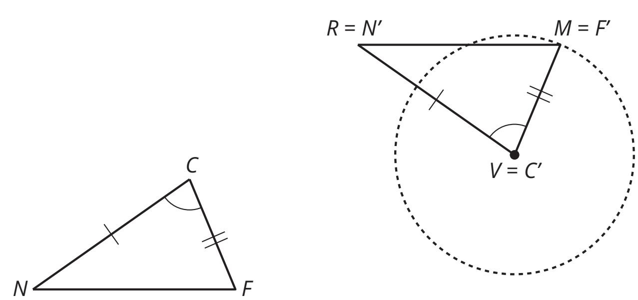
You know that you can map $\overline{CN}$ to $\overline{VR}$ in one or two reflections.



Since reflections preserve distance, this means that point F' must be on the circle centered at V with radius $\overline{VM}$, because $\overline{CF} \cong \overline{VM}$.



And since reflections preserve angle measures, there are only two possible locations for point F' to be on the circle. If point F' is not at M, then a reflection across $\overline{VR}$ will map point F' onto point M, $R_{\overline{VR}}(F') \rightarrow M$.



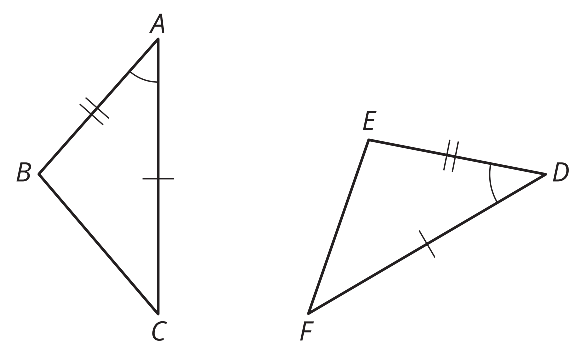
1. Explain the final steps in the Worked Example.

a. Why are there only two locations that point F' could be in relation to points V and R?

b. Why will a reflection across $\overline{VR}$ map point F' onto point M?

Because this relationship has been proved to be true, you may now refer to it as a theorem. The **Side-Angle-Side Congruence Theorem (SAS)** states: "If two sides and the included angle of one triangle are congruent to the corresponding two sides and the included angle of a second triangle, then the triangles are congruent."

2. Consider the diagram shown where $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\angle A \cong \angle D$.
Prove $\triangle ABC \cong \triangle DEF$. Explain your steps.



3. Write congruence statements for all corresponding parts and angle relationships of the pre-image and the image.

ACTIVITY

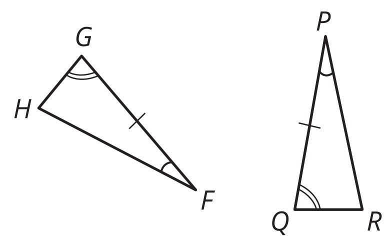
2.4

Angle-Side-Angle Congruence

An **included side** is the side between two angles of a triangle.

Let's consider two triangles that have two angles and an included side congruent. Use this criteria to prove the two triangles are congruent. Use reasoning similar to that used for the Side-Angle-Side Theorem.

1. Consider $\triangle FGH$ and $\triangle PQR$ where $\overline{FG} \cong \overline{PQ}$, $\angle G \cong \angle Q$, and $\angle F \cong \angle P$. Prove that $\triangle FGH \cong \triangle PQR$.



Because this relationship has been proved to be true, you may now refer to it as a theorem. The **Angle-Side-Angle Congruence Theorem (ASA)** states: "If two angles and the included side of one triangle are congruent to the corresponding two angles and the included side of another triangle, then the triangles are congruent."

2. Write congruence statements for all corresponding parts and angle relationships of the pre-image and the image.

ACTIVITY 2.5

Non-Examples of Congruence Theorems

Thus far, you have explored and proven each of the triangle congruence theorems:

- Side-Side-Side Congruence Theorem (SSS)
- Side-Angle-Side Congruence Theorem (SAS)
- Angle-Side-Angle Congruence Theorem (ASA)

1. Juno wondered why AAA isn't on the list of congruence theorems. Provide a counterexample to show Juno why Angle-Angle-Angle (AAA) *is not* considered a triangle congruence theorem.

2. Juno also wondered why SSA isn't on the list of congruence theorems. Provide a counterexample to show Juno why Side-Side-Angle (SSA) *is not* considered a triangle congruence theorem.

TALK the TALK

The Right Combination

1. **Complete the graphic organizer to summarize what you have learned about the triangle congruence theorems.**