2

ASA, SAS, and SSS

Proving Triangle Congruence Theorems

MATERIALS

Compasses
Patty Paper
Protractors
Rulers
Straightedges

Lesson Overview

Students use what they have learned in the previous topic: (1) isometries preserve distances and angle measures, (2) any point in the plane can be reflected across a line to map to another point in the plane, and (3) a point is equidistant from two other points if and only if it lies on their perpendicular bisector. They use these facts to create and verify proofs of the SSS, SAS, and ASA Congruence Theorems using rigid motion transformations. Students then explore some non-examples of congruence theorems (AAA and SSA).

Geometry
Proof and Congruence

(G.6) The student uses the process skills with deductive reasoning to prove and apply theorems by using a variety of methods such as coordinate, transformational, and axiomatic and formats such as two-column, paragraph, and flow chart. The student is expected to:

(B) prove two triangles are congruent by applying the Side-Angle-Side, Angle-Side-Angle, Side-Side-Side, Angle-Angle-Side, and Hypotenuse-Leg congruence conditions.

(C) apply the definition of congruence, in terms of rigid transformations, to identify congruent figures and their corresponding sides and angles.

ELPS

1.A, 1.C, 1.E, 1.F, 1.G, 2.C, 2.E, 2.I, 3.D, 3.E, 4.B, 4.C, 4.D, 4.J, 5.B, 5.F, 5.G

Essential Ideas

- A proof is a series of statements and corresponding reasons forming a valid argument that starts with a hypothesis and arrives at a conclusion.
- The Side-Side-Side (SSS) Congruence Theorem states: "If three sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent."
- Corresponding parts of congruent triangles are congruent, abbreviated as CPCTC, is often used as a reason for stating congruences in geometric proofs after triangles have been proven congruent.

- The The Side-Angle-Side (SAS) Congruence Theorem states: "If two sides and the included angle of one triangle are congruent to the corresponding sides and included angle of another triangle, then the triangles are congruent."
- The Angle-Side-Angle (ASA) Congruence Theorem states: "If two angles and the included side of one triangle are congruent to the corresponding angles and included side of another triangle, then the triangles are congruent."
- Triangle congruence theorems such as SSS, SAS, and ASA can be proven using rigid motion transformations.

Lesson Structure and Pacing: 3 Days

Day 1

Engage

Getting Started: Necessary Conditions

Students use measuring tools to determine whether two triangles are congruent. They conclude that measuring all six corresponding parts of both triangles may not be necessary to demonstrate two triangles are congruent.

Develop

Activity 2.1: Congruent Line Segments by Reflection

Students use the appropriate notation to mark segments and angles congruent. A Worked Example uses transformations to prove that a segment can be mapped onto itself in at most two reflections. This mathematical concept is used to prove triangle congruence theorems in the remainder of this lesson.

Day 2

Activity 2.2: Side-Side-Side Congruence

Students use reflections to prove the Side-Side-Side Congruence Theorem for triangles. They then using the reasoning that corresponding parts of congruent triangles are congruent (CPCTC) to write equality and congruence statements for all corresponding parts of the triangles.

Activity 2.3: Side-Angle-Side Congruence

The Side-Angle-Side Congruence Theorem is stated. Students analyze a Worked Example that uses rigid motion transformations to prove this theorem. They explain the steps of the proof, apply the proof to a different situation, and use CPCTC to write congruence statements for all corresponding parts.

Day 3

Activity 2.4: Angle-Side-Angle Congruence

Students prove the Angle-Side-Angle Congruence Theorem using reflections similar to those they used in the proof of SAS. They also use CPCTC to write congruence statements for all corresponding parts.

Activity 2.5: Non-Examples of Congruence Theorems

Students create counterexamples to show why Angle-Angle-Angle is not considered a valid congruence theorem and why Side-Side-Angle is not considered a valid congruence theorem.

Demonstrate

Talk the Talk: The Right Combination

Students complete a graphic organizer to summarize what they have learned about the triangle congruence theorems.

Getting Started: Necessary Conditions

Facilitation Notes

In this activity, students use measuring tools to determine whether two triangles are congruent. They conclude that measuring all six corresponding parts of both triangles may not be necessary to demonstrate two triangles are congruent.

Have the students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

As students work, look for

- Comments regarding experimenting with triangles in middle school, when they determined whether a unique triangle could be constructed from a combination of three measures of angles and/or sides.
- The realization that if two pairs of corresponding angles in a triangle are congruent, then the third pair of corresponding angles are also congruent.

Misconception

Students may incorrectly extend what they remember about SSS to include AAA. If this is the case, ask students to provide a counterexample. One such counterexample is two different-sized isosceles right triangles.

Questions to ask

- What unit of measure did you use to measure the sides of each triangle?
- What side of $\triangle RNL$ is congruent to side $\overline{AB}$? Side $\overline{AC}$? Side $\overline{BC}$?
- What angle of $\triangle RNL$ is congruent to $\angle A$? $\angle B$? $\angle C$?
- Do you think you need to measure all six angles and sides of two triangles to determine whether the triangles are congruent? Why or why not?
- Do you think measuring only 5 pairs of corresponding sides and angles would be sufficient to prove the triangles are congruent? What about 4 pairs of corresponding sides and angles? What about 3 pairs of corresponding sides and angles?
- Is there a point in your measuring that you knew you could stop because you knew the remainder of the sides and angles would have to be congruent as well? If so, when?

Summary

If two triangles are congruent, three pairs of corresponding sides are equal in length and three pairs of corresponding angles are equal in measure. It is possible to conclude two triangles are congruent using fewer than six pairs of corresponding parts.

Activity 2.1
Congruent Line Segments by Reflection

DEVELOP

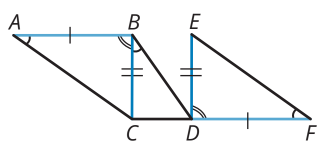
Facilitation Notes

In this activity, students use the appropriate notation to mark segments and angles congruent. A Worked Example uses transformations to prove that a segment can be mapped onto itself in at most two reflections. This mathematical concept is used to prove triangle congruence theorems in the remainder of this lesson.

Ask a student to read the introduction aloud. Analyze the Worked Example as a class.

Differentiation strategy

To assist all students in analyzing the Worked Example, suggest they use colored pencils to trace congruent sides and mark congruent angles. Besides helping to understand the new notation, some students may benefit by making this a standard practice in interpreting labeled diagrams.



Misconception

In the Worked Example, $\overline{BC}$ and $\overline{ED}$ appear to be parallel. Students may confuse the slash markers with the arrow notation to note parallel lines. Clarify this confusion if it arises.

Questions to ask

- Why does it make sense that the congruence statement is written as $\overline{AB} \cong \overline{FD}$ rather than $\overline{AB} \cong \overline{DF}$?
- How would you complete the congruence statement $\overline{ED} \cong$ ________ ?
- How do the corresponding parts of the triangles help you determine the order to name the endpoints of the line segments?

Have the students work with a partner or in a group to complete Question 1. Share responses as a class.

Misconception

Students may think all angles can be named using only their vertex point if there are markers on the diagram denoting angles of equal measure. Angles can be named using one or three letters; however, when there is more than one angle sharing the same vertex, three letters must be used to avoid confusion.

Questions to ask

- How do you know that $\angle R \cong \angle W$?
- Why is it acceptable to name $\angle R$ and $\angle W$ with single letters?
- Why do the congruent angles with vertices at points X and T need to be named with three letters rather than one letter?
- What is another way to label each of the angles with a single arc marker?

Differentiation strategies

To extend the lesson, have students further analyze the diagrams in Question 1.

- Using Question 1, part (a), demonstrate the usefulness of decomposing figures.
  - Demonstrate how to decompose the diagram into two triangles, $\triangle RST$ and $\triangle WBX$.
  - Discuss why $\angle S \cong \angle B$, and how this may not have been as apparent without decomposing the diagram. If two angles of one triangle are congruent to two angles of a second triangle, then the third angles of each triangle must also be congruent.
- Using Question 1, part (b), demonstrate the usefulness of extending diagrams.
  - Discuss why $\overline{DM}$ is parallel to $\overline{ZC}$. Have students extend line segments $\overline{DM}$, $\overline{ZC}$, and $\overline{TZ}$ to see that corresponding angles are congruent. Because corresponding angles are congruent, the lines are parallel.
  - Have students use the appropriate notation to mark the parallel lines.

Ask a student to read the information following Question 1 aloud. Analyze the Worked Example as a class.

Differentiation strategies

To assist all students,

- Have them highlight the three statements that they can use as facts to prove a conjecture. Make it explicit while analyzing the Worked Example how these exact phrases were used to reason through the proof. It may be worthwhile to have students highlight these phrases in the Worked Example, as well.
- Remind them that they explored this concept with Sunita's conjecture in Lesson 3 in the previous topic. Now they will concentrate on using transformations to write a formal proof for Sunita's conjecture.
- Model and summarize the procedure using patty paper prior to analyzing the Worked Example. Summarize the two steps: (1) reflect a point to a point, and (2) reflect a point to a point (and

this time, the result is that the entire segment is also mapped onto the congruent figure). Use this same terminology when analyzing the Worked Example.

- Have students interact with the Worked Example.
  - Mark congruent segments by using slash markers.
  - Construct line *m* in the first diagram, then compare their work with the second diagram. Label point *C* also as *A'*. Relate this to step 1 using patty paper.
  - Write the congruence statements vertically to see the use of the transitive property.

$$\overline{CB'} \cong \overline{CD}$$
$$\overline{AB'} \cong \overline{CD}$$
$$\overline{AB'} \cong \overline{CB'}$$

  - Construct line *n* themselves in the middle diagram, then compare their work with the final diagram. Label point *D* also as *B'*. Relate this to step 2 using patty paper.
- After analyzing the Worked Example, have students highlight "a segment can be mapped onto itself in at most two reflections" in the prove-statement of the Worked Example. Make it explicit that now that this statement has been proven, it can be used as a reason in future geometric proofs along with the three other highlighted statements.

Questions to ask

- How are a counterexample and proof alike? How are they different?
- Does it matter whether point *A* is mapped to point *C* or point *D*? Explain.
- What is the result of the reflection across line *m*?
- What is another way to say that point *C* is equidistant from point *B'* and point *D*?
- Why is it important to know that point *C* is the same distance from points *B'* and *D*?
- Could you have constructed an angle bisector through point *C* rather than a perpendicular bisector through $\overline{B'D}$? Why would this also work?
- What is the result of the reflection across line *n*?
- How were the three facts listed before the Worked Example used in the proof?

Have the students work with a partner or in a group to complete Questions 2 and 3. Share responses as a class.

Differentiation strategy

To scaffold support in creating an example requiring just one reflection, provide a hint by asking how students could eliminate the first creation of a perpendicular bisector in the Worked Example. The answer is that they could already have the point of one segment mapped to a point of the other segment.

Questions to ask

- Why does having point *A* and point *C* at the same location eliminate one reflection?
- Would having point *B* and point *D* at the same location have the same results? Why or why not?
- How could you create an example using parallel lines?

Summary

A conjecture can be proven using known facts in a logical order.

Activity 2.2

Side-Side-Side Congruence

Facilitation Notes

In this activity, students use reflections to prove the Side-Side-Side Congruence Theorem for triangles. They then using the reasoning that corresponding parts of congruent triangles are congruent (CPCTC) to write equality and congruence statements for all corresponding parts of the triangles.

Ask a student to read the introduction and theorem aloud. Discuss as a class.

As students work, look for

Prior knowledge that a unique triangle can be constructed from a combination of three side lengths (as long as the side lengths comply with the Triangle Inequality Theorem).

Questions to ask

- Can any three segments be connected to form a triangle? Why not?
- What relationship must exist among the segment lengths to know whether or not a triangle can be formed?
- Given three segments, when you connect them, will you always get the same triangle? Why is this true for triangles, but not quadrilaterals?
- Consider the statement, "Given three segments, a unique triangle can be constructed." What is meant by the term *unique*?

Have the students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

Differentiation strategy

To scaffold support, suggest that students use patty paper instead of, or prior to, using a compass and straightedge to demonstrate triangle congruence.

As students work, look for

- The construction of two perpendicular bisectors.
- Appropriate labeling of vertices after each reflection.

Misconception

Students may think a point cannot be reflected onto an existing point and be named differently. After the first reflection, point V' and point B are at the same location. After the second reflection, points B, V', and V'' are at the same location, points A'' and K are at the same location, and points R'' and F are at the same location.

Questions to ask

- Explain your steps to map point V onto point B.
- How did you label the reflected triangle?
- What reason did you select from the original three facts provided?
- How do you know that $\triangle VAR \cong \triangle V'A'R'$?
- Could you have chosen a different line segment to reflect? Why or why not?
- Explain your steps to map $\overline{V'A'}$ onto $\overline{B'K'}$.
- Identify the locations being named by more than one point.
- Why are you able to use a reason based upon a reflected segment rather than a reflected point?
- What is special about the pair of triangles so that the two reflections map all three sides of one triangle onto the other triangle?
- What do reflections have to do with knowing that $\triangle V'A'R' \cong \triangle BKF$?
- What do reflections have to do with knowing that $\triangle VAR \cong \triangle BKF$?
- If 3 corresponding sides of two triangles are congruent, can the triangles always be mapped onto each other by a series of reflections?
- If the size and shape of each of the 3 triangles is the same, are the pairs of corresponding sides and angles congruent?
- If the pairs of corresponding sides and angles are congruent, are the triangles congruent?

Have the students work with a partner or in a group to read the information after Question 2 and complete Question 3. Share responses as a class.

Differentiation strategies

- To assist all students,
  - Have them highlight the SSS Theorem and CPCTC as reference for reasons that can be used in proofs.

   - Direct them to arc markers to indicate the congruent angle pairs. Although not a requirement, for this first time it may help students to understand the power of CPCTC.
- To scaffold support, demonstrate how to use the placement of the letters in the triangle congruence statement to write equality and congruence statements for corresponding parts.

As students work, look for

The use of the diagram or the triangle congruence statement to list equality and congruent statements for all corresponding parts of the two triangles.

Misconceptions

- Because $\overline{VA}$ and $\overline{AV}$ are names for the same line segment, students may think these representations are interchangeable when writing congruence statements. The order in which the letters appear when naming sides, angles, and triangles in a congruence statement is important. The congruence statement $\overline{VA} \cong \overline{BK}$ implies point V corresponds to point B and point A corresponds to point K. For this reason, address the inconsistency in a statement such as $\overline{AV} \cong \overline{BK}$.
- Because students have become accustomed to writing equality and congruence statements, they may not realize that the congruent angle pairs are the result of using the SSS Congruence Theorem to first establish $\triangle VAR \cong \triangle BKF$, then applying CPCTC. Consider the use of a flow chart to demonstrate clearly the order of logic applied.

Questions to ask

- What does the abbreviation CPCTC stand for?
- Explain what CPCTC means in your own words.
- How many corresponding parts are used to prove $\triangle VAR$ and $\triangle BKF$ are congruent? What were they?
- How do you know that the three pairs of corresponding angles are also congruent?
- Did you use the diagram or the congruence statement to write your equality and congruence statements?
- How can the congruence statement be used to write the equality and congruence statements?
- Which method do you prefer?

Differentiation strategy

To extend the activity, demonstrate how to construct a triangle using a compass and straightedge given three side lengths. Have students duplicate the process using patty paper, then compare their sheets of patty paper to demonstrate that the same angles were formed each time.

Summary

The Side-Side-Side Congruence Theorem (SSS) states: "If three sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent." Once two triangles are proven congruent, it can be stated that all corresponding sides and angles in the triangles are also congruent because corresponding parts of congruent triangles are congruent (CPCTC).

Activity 2.3

Side-Angle-Side Congruence

Facilitation Notes

In this activity, the Side-Angle-Side Congruence Theorem is stated. Students analyze a Worked Example that uses rigid motion transformations to prove this theorem. They explain the steps of the proof, apply the proof to a different situation, and use CPCTC to write congruence statements for all corresponding parts.

Ask a student to read the definition and theorem aloud. Analyze the Worked Example as a class.

Differentiation strategy

To assist all students, complete the proof as a class. Perform each step of the construction as the proof develops, talking about why the next step is necessary and valid. Use colored pencils to emphasize different aspects of the developing construction. Students may have a better understanding of the mathematical reasoning as it unfolds if each step is discussed as it is drawn.

Have the students work with a partner or in a group to complete Question 1. Share responses as a class.

Misconception

Students may not understand why the location for point F' was chosen out of the two possibilities. Remind them to look at the original sketch.

Questions to ask

- Why is it acceptable to use reflections? Do they change the measures of the angles and sides?
- What is the connection between the possible locations of the reflection of point *F* and congruent corresponding angles?
- Why do you think a circle was included in the diagram?
- What is the relationship between $m\angle C$ and $m\angle NCF$?

- What is the relationship between $\overline{CF}$ and $\overline{VM}$?
- How are the reflection of point F and the length of $\overline{CF}$ related?
- If point M is reflected across $\overline{VR}$, will the reflection of point M also be located on circle V? How do you know?
- Does point V lie on the perpendicular bisector of $\overline{FM}$? How do you know?
- Is $\overline{VR}$ the perpendicular bisector of $\overline{FM}$? How do you know?

Have the students work with a partner or in a group to complete Questions 2 and 3. Share responses as a class.

Questions to ask

- Outline your strategy to prove the triangles are congruent.
- Is there another strategy that works? If so, what is it?
- Why is it acceptable to use reflections?
- How did you know what side to map onto the other? Does it make a difference?
- Why is it important to construct a circle?
- Which vertex did you use to construct a circle?
- What side of $\triangle DEF$ are you using to establish the radius of circle D?
- Why is $\overline{BC} \cong \overline{EF}$?
- How did you determine the other congruence statements?

Summary

The Side-Angle-Side Congruence Theorem (SAS) states: "If two sides and the included angle of one triangle are congruent to the corresponding sides and included angle of another triangle, then the triangles are congruent." This theorem can be proven using rigid motion transformations.

Activity 2.4

Angle-Side-Angle Congruence

Facilitation Notes

In this activity, students prove the Angle-Side-Angle Congruence Theorem using reflections similar to those they used in the proof of SAS. They also use CPCTC to write congruence statements for all corresponding parts.

Have the students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

As students work, look for

The center point of the circle used for the reflection of $\triangle FGH$. Did students use point Q for the center point and $\overline{QR}$ as the perpendicular bisector of the segment formed by mapping the two possible locations for the reflection of point F?

Questions to ask

- Can you map $\overline{FG}$ onto $\overline{PQ}$ in two reflections?
- Which vertex in ΔPQR are you using to construct a circle?
- Does point G map onto point Q?
- What side of ΔFGH are you using to establish the radius of circle Q?
- Can point F be reflected onto circle Q in two possible locations? Is one of these locations on top of point P?
- Why must the radius of the circle be the length of $\overline{PQ}$?
- Is point R equidistant from points P and the reflection of point $F(F')$?
- Does point R lie on the perpendicular bisector of $(\overline{PF'})$?
- Is $\overline{QR}$ the perpendicular bisector of $(\overline{PF'})$? How do you know?
- Does $R_{\overline{QR}}(F) = P$?
- What does CPCTC mean?
- How did you apply CPCTC?

Summary

The Angle-Side-Angle Congruence (ASA) states: "If two angles and the included side of one triangle are congruent to the corresponding angles and included side of another triangle, then the triangles are congruent." This theorem can be proven using rigid motion transformations.

Activity 2.5

Non-Examples of Congruence Theorems

Facilitation Notes

In this activity, students create counterexamples to show why Angle-Angle-Angle is not considered a valid congruence theorem and why Side-Side-Angle is not considered a valid congruence theorem.

Have the students work with a partner or in a group to complete Questions 1 and 2. Share responses as a class.

As students work, look for

Different counterexamples, using equilateral triangles, right triangles, isosceles triangles, and scalene triangles.

Misconception

Students may think they learned previously that AAA is valid reasoning to prove triangles congruent. If so, remind them of the difference between the concepts of similarity and congruence and how their counterexamples relate to similarity.

Differentiation strategy

To scaffold support, suggest that students use patty paper to create their counterexamples. In each case, provide them the angle and side measures and rephrase the question by asking them to create two different triangles using the same measures. Then, discuss how the two different triangles are a counterexample.

Questions to ask

- How did you determine what angles to use in your counterexample?
- What relationship exists between the two triangles you formed using AAA?
- How many different triangles can you form using SSA?
- Why does the creation of two different triangles disprove the triangle congruence conjecture?
- How many counterexamples are necessary to disprove any conjecture?

Summary

The congruence conjectures AAA and SSA are not valid because you can create counterexamples to disprove them.

DEMONSTRATE

Talk the Talk: The Right Combination

Facilitation Notes

In this activity, students complete a graphic organizer to summarize what they have learned about the triangle congruence theorems.

Have the students work with a partner to complete the graphic organizer. Share responses as a class.

As students work, look for

Diagrams with labels, explanations in words, and congruence statements.

Questions to ask

- What different representations did you use?
- State the theorem in your own words.
- How did you use diagrams to support your thinking?
- Explain how you used congruence statements to explain the diagrams.
- Why isn't it necessary to prove all six parts of a triangle are congruent to prove they are congruent?

Summary

Triangle congruence theorems are methods to prove two triangles are congruent using three corresponding parts of each triangle, rather than all six parts of each triangle. The triangle congruence theorems currently proven are the SSS (Side-Side-Side), SAS (Side-Angle-Side), and ASA (Angle-Side-Angle) Congruence Theorems. The triangle congruence conjectures proven to be false are AAA (Angle-Angle-Angle) and SSA (Side-Side-Angle).

NOTES

2

ASA, SAS, and SSS

Proving Triangle Congruence Theorems

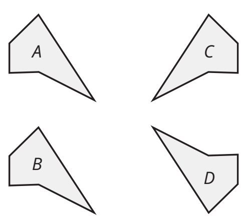
Warm Up

Describe transformations that map Figure *A* onto the other figures shown.

1. Figure *B*

2. Figure *C*

3. Figure *D*



Learning Goals

- Use the definition of congruence in terms of rigid motions to show that two triangles are congruent.
- Prove the Side-Side-Side Congruence Theorem using rigid motion transformations.
- Prove the Side-Angle-Side Congruence Theorem using rigid motion transformations.
- Prove the Angle-Side-Angle Congruence Theorem using rigid motion transformations.

Key Terms

- Side-Side-Side Congruence Theorem (SSS)
- corresponding parts of congruent triangles are congruent (CPCTC)
- Side-Angle-Side Congruence Theorem (SAS)
- included angle
- Angle-Side-Angle Congruence Theorem (ASA)
- included side

You have defined the transformations that produce isometries. How can you use isometries to prove congruence theorems?

Warm Up Answers

1. Translate Figure *A* down onto Figure *B*.
2. Reflect Figure *A* across a vertical line of reflection between Figures *A* and *C*.
3. Rotate Figure *A* counterclockwise 180° onto Figure *D*.

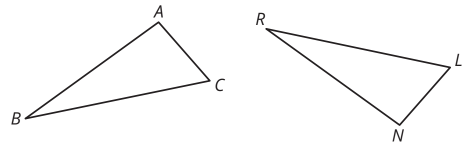
Answers

1. I measured all three pairs of corresponding angles, and each pair of angles was congruent. I measured all three pairs of corresponding sides, and each pair of sides was congruent.
2. Answers will vary. Student answers may range between comparing 3 to 6 pairs of corresponding parts, requiring 6 to 12 measurements.

GETTING STARTED

Necessary Conditions

Consider the two triangles shown.



Each triangle pair has 6 relationships—3 pairs of sides and 3 pairs of angles. If the two triangles are congruent, all the corresponding side lengths and all the corresponding angle measures must be equal.

1. Use a ruler and protractor to determine whether the two triangles are congruent. Explain your strategy.

2. What is the minimum number of measurements you could make to determine whether the two triangles are congruent? Explain your reasoning.

ACTIVITY 2.1

Congruent Line Segments by Reflection

Congruent line segments and congruent angles are often denoted using special markers, rather than given measurements.

Slash markers can be used to indicate congruent line segments. When multiple line segments contain a single slash marker, this implies that all of those line segments are congruent. Double and triple slash markers can also be used to denote other line segment congruencies.

Arc markers can be used to indicate congruent angles. When multiple angles contain a single arc marker, this implies that those angles are congruent. Double and triple arc markers can also be used to denote other angle congruencies.

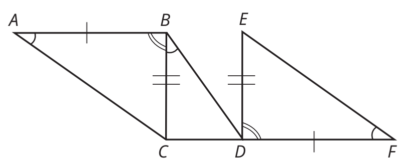
Worked Example

The markers on the diagram indicate congruent line segments.

$\overline{AB} \cong \overline{FD}$ and $\overline{BC} \cong \overline{DE}$

$\angle BAC \cong \angle CBD \cong \angle DFE$

$\angle ABC \cong \angle FDE$

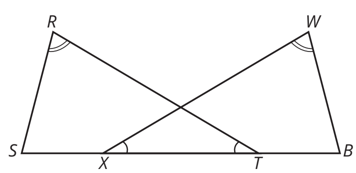


Think about:

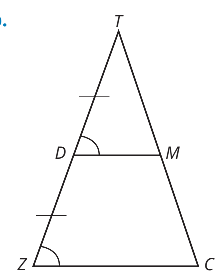
Although $\overline{DF}$ and $\overline{FD}$ represent the same line segment, when writing a congruence statement for segments that are part of a larger figure, think about how the sides and angles in the figure correspond to one another.

1. Write the congruence statements represented by the markers in each diagram.

a.



b.



Think about:

Make sure you are properly naming angles.

Answers

1a. $\angle R \cong \angle W$
$\angle WXB \cong \angle RTS$

1b. $\angle TDM \cong \angle TZC$
$\overline{TD} \cong \overline{DZ}$

ELL Tip

Two non-mathematical terms that appear in this activity are *implies* and *denotes*. Define *imply* as *to strongly suggest something without proof*. Define *denote* as *to stand as a name or symbol for something*. Synonyms for *imply* include *indicate, infer, signal,* and *insinuate*. Synonyms for *denote* include *indicate, symbolize, represent, mean,* and *designate*. Read through the beginning paragraphs of the activity and discuss the terms *implies* and *denotes* as they are used in the context of the activity.

Remember:

A proof is a series of statements and corresponding reasons forming a valid argument that starts with a hypothesis and arrives at a conclusion.

In previous lessons, you learned that:

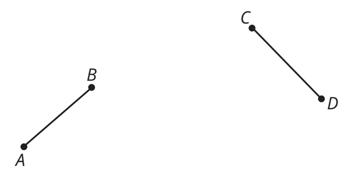
(1) Isometries preserve distances and angle measures.
(2) Any point in the plane can be reflected across a line to map onto another point in the plane.
(3) A point is equidistant from two other points if and only if it lies on their perpendicular bisector.

You can use these facts to prove a conjecture that you have explored. Prove that you can always map one congruent line segment onto the other using at most two reflections.

Worked Example

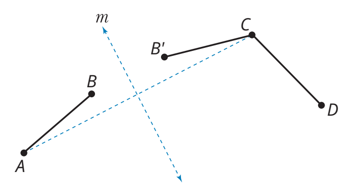
Prove that a segment can be mapped onto itself in at most two reflections.

Suppose that $\overline{AB} \cong \overline{CD}$.



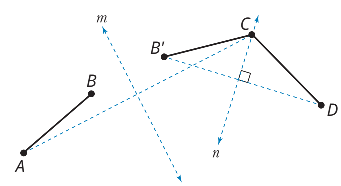
Since any point in the plane can be reflected across a line to map onto another point in the plane, you know that point *C* is a reflection of point *A* across line *m*. Reflect $\overline{AB}$ across line *m*, $R_m(\overline{AB}) \rightarrow (\overline{CB'})$.

Reflections preserve distances, so you know that $\overline{CB'} \cong \overline{CD}$, because $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{CB'}$. Thus, point *C* is equidistant from point *B'* and from point *D*.

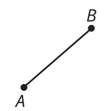


Since a point is equidistant from two other points if and only if it lies on their perpendicular bisector, this means that point *C* lies on the perpendicular bisector of $\overline{B'D}$ (line *n*).

Thus, one last reflection across the perpendicular bisector maps $\overline{CB'}$ onto $\overline{CD}$, $R_n(\overline{CB'}) \rightarrow (\overline{CD})$.



2. **The proof in the Worked Example shows two reflections. Create an example in which $\overline{AB}$ maps onto $\overline{CD}$ in just one reflection. Explain your example.**

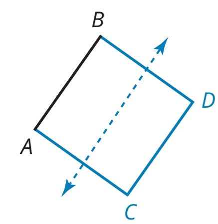


3. **Use the Worked Example to explain why you do not need more than two reflections to map a line segment onto a congruent line segment in the plane.**

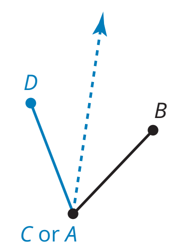
3. One reflection is needed to map an endpoint of one segment onto the endpoint of the other segment. The second reflection maps the entire segment onto the congruent line segment.

Answers

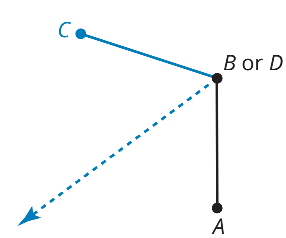
2. Sample answers.
$\overline{CD}$ and $\overline{AB}$ could be opposite sides of a square, such that point *A* and point *C* are consecutive vertices. The line of reflection would be the line connecting the midpoint of sides $\overline{AC}$ and $\overline{BD}$.



$\overline{CD}$ could be the same length as $\overline{AB}$, and point *C* and point *A* could be at the same location. The line of reflection would be the angle bisector of $\angle DAB$ or $\angle DCB$.

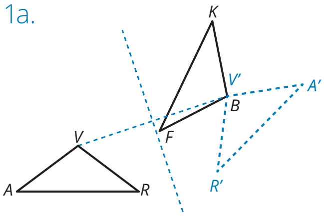


$\overline{CD}$ could be the same length as $\overline{AB}$, and point *B* and point *D* could be at the same location. The line of reflection would be the angle bisector of $\angle ABC$ or $\angle ADC$.



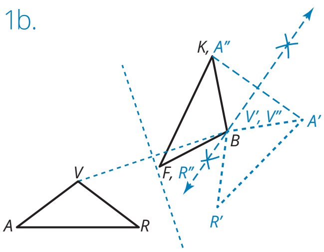
Answers

1a.



Any point in the plane can be reflected across a line to map onto another point in the plane. In this case, *V* is mapped onto *B*.

1b.



A segment can be mapped onto itself in at most two reflections.

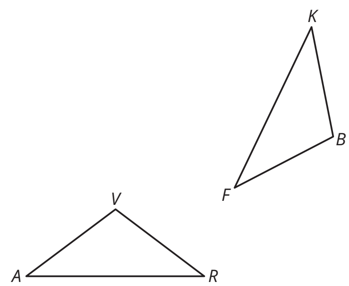
ACTIVITY **2.2** Side-Side-Side Congruence

In previous courses, you investigated the conditions necessary for forming a triangle with different side lengths. Now you will prove that triangles are congruent given different minimal criteria.

> You can use the fact that a segment can be mapped onto itself in at most two reflections to prove $\triangle VAR$ is congruent to $\triangle BKF$.

Consider two triangles such that the three sides of one triangle are congruent to the three sides of the second triangle. You can prove that this criteria is sufficient to demonstrate the two triangles are congruent.

Given $\triangle VAR$ and $\triangle BKF$. Suppose $VA = BK$, $VR = BF$, and $AR = KF$.



1. Complete the steps to show the proof that $\triangle VAR \cong \triangle BKF$.

a. Draw the reflection of $\triangle VAR$ across a line that maps point *V* onto point *B*. Label the image as $\triangle V'A'R'$. Give the reason(s) you can create this reflection.

b. Draw the reflection of $\triangle V'A'R'$ which maps $\overline{V'A'}$ onto $\overline{BK}$. Label the image as $\triangle V''A''R''$. Give the reason(s) you can create this reflection.

ELL Tip

Read aloud the second sentence in the activity, "Now you will prove that triangles are congruent given different *minimal criteria*." Explain the phrase *minimal criteria* by breaking it down. Define *minimal* as *the smallest amount or degree of something*. Define *criteria* as *principles or standards by which something is judged or decided*. Discuss the context of the phrase in the sentence, and explain that proving triangles congruent using *minimal criteria* means to use the least amount of information possible about the corresponding angles and sides of the triangles in order to prove that they are congruent.

2. **Summarize the proof you completed in Question 1. Explain how knowing that the three corresponding sides of two triangles are congruent proves that the two triangles are congruent.**

Because this relationship has been proved to be true, you may now refer to it as a theorem. The **Side-Side-Side Congruence Theorem (SSS)** states: "If three sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent."

Any theorem can be used in the future as a reason in other proofs.

If two triangles are congruent, then each part of one triangle is congruent to the corresponding part of the other triangle. **Corresponding parts of congruent triangles are congruent**, abbreviated as **CPCTC**, is often used as a reason in proofs. CPCTC states that corresponding angles and sides in two congruent triangles are congruent. This reason can only be used after you have proven that the triangles are congruent.

3. **Write congruence statements for all corresponding side and angle relationships of the pre-image and the image.**

Answers

2. Sample answer. One side of $\triangle VAR$ is mapped onto its corresponding side in $\triangle BKF$ through two reflections. This series of reflections, which preserves distance and angle measures, also maps the remainder of the triangle, its other two pairs of corresponding sides and three angles, onto each other. Knowing that two triangles have three pairs of corresponding sides proves the two triangles are congruent.

3. $VA = BK$
   $VR = BF$
   $AR = KF$

   $m\angle V = m\angle B$
   $m\angle A = m\angle K$
   $m\angle R = m\angle F$

   $\overline{VA} \cong \overline{BK}$
   $\overline{VR} \cong \overline{BF}$
   $\overline{AR} \cong \overline{KF}$

   $\angle V \cong \angle B$
   $\angle A \cong \angle K$
   $\angle R \cong \angle F$

ACTIVITY 2.3

Side-Angle-Side Congruence

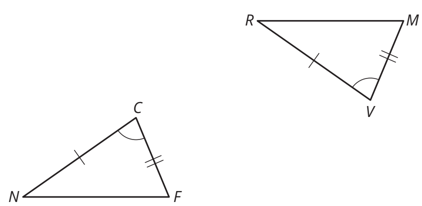
An **included angle** is the angle formed by two sides of a triangle.

Consider two triangles that have two sides and an included angle congruent. Analyze the proof using rigid motion transformations to demonstrate that this criteria is sufficient to prove the two triangles are congruent.

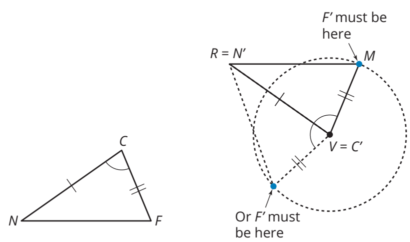
Worked Example

Given $\triangle CNF$ and $\triangle VRM$, $\overline{CN} \cong \overline{VR}$, $\overline{CF} \cong \overline{VM}$, and $\angle C \cong \angle V$. Prove $\triangle CNF \cong \triangle VRM$.

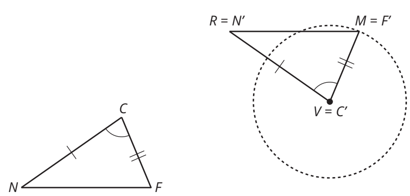
You know that you can map $\overline{CN}$ to $\overline{VR}$ in one or two reflections.



Since reflections preserve distance, this means that point F' must be on the circle centered at V with radius $\overline{VM}$, because $\overline{CF} \cong \overline{VM}$.



And since reflections preserve angle measures, there are only two possible locations for point F' to be on the circle. If point F' is not at M, then a reflection across $\overline{VR}$ will map point F' onto point M, $R_{\overline{VR}}(F') \rightarrow M$.



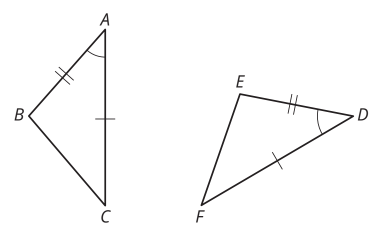
1. Explain the final steps in the Worked Example.

a. Why are there only two locations that point *F′* could be in relation to points *V* and *R*?

b. Why will a reflection across $\overline{VR}$ map point *F′* onto point *M*?

Because this relationship has been proved to be true, you may now refer to it as a theorem. The **Side-Angle-Side Congruence Theorem (SAS)** states: "If two sides and the included angle of one triangle are congruent to the corresponding two sides and the included angle of a second triangle, then the triangles are congruent."

2. Consider the diagram shown where $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\angle A \cong \angle D$.
Prove $\triangle ABC \cong \triangle DEF$. Explain your steps.



3. Write congruence statements for all corresponding parts and angle relationships of the pre-image and the image.

LESSON 2: ASA, SAS, and SSS • 9

Answers

1a. The congruent corresponding angle measures tell us that point *F′* can be at only 2 places on the circle—as a reflection across side *VR*.

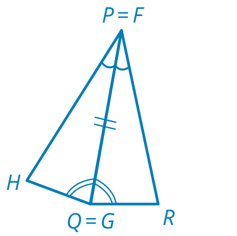
1b. Point *M* and point *F* are equidistant from point *V*, so *V* lies on the perpendicular bisector of $\overline{MF'}$.

2. $\overline{AC}$ maps onto $\overline{DF}$ in one or two reflections. Reflections preserve distance, so point *B′* must be on a circle centered at point *D* with a radius of *DE*, because $\overline{AB} \cong \overline{DE}$. Reflections preserve angle measure, so there are only two possible locations for point *B′* to be on the circle. If point *B′* is not at point *E*, then a reflection across $\overline{DF}$ will map point *B′* onto point *E*, $R_{\overline{DF}}(B') = E$.

3. $\overline{AB} \cong \overline{DE}$
$\overline{AC} \cong \overline{DF}$
$\overline{BC} \cong \overline{EF}$

$\angle A \cong \angle D$
$\angle B \cong \angle E$
$\angle C \cong \angle F$

Answers

1. 

Side $\overline{FG}$ can be mapped onto side $\overline{PQ}$ in at most two reflections. A final reflection maps the two triangles onto each other.

2. $\angle H \cong \angle R$
$\angle G \cong \angle Q$
$\angle F \cong \angle P$
$\overline{HG} \cong \overline{RQ}$
$\overline{HF} \cong \overline{RP}$
$\overline{GF} \cong \overline{QP}$

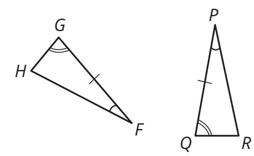
ACTIVITY **2.4**

Angle-Side-Angle Congruence

An **included side** is the side between two angles of a triangle.

Let's consider two triangles that have two angles and an included side congruent. Use this criteria to prove the two triangles are congruent. Use reasoning similar to that used for the Side-Angle-Side Theorem.

1. Consider $\triangle FGH$ and $\triangle PQR$ where $\overline{FG} \cong \overline{PQ}$, $\angle G \cong \angle Q$, and $\angle F \cong \angle P$. Prove that $\triangle FGH \cong \triangle PQR$.



Because this relationship has been proved to be true, you may now refer to it as a theorem. The **Angle-Side-Angle Congruence Theorem (ASA)** states: "If two angles and the included side of one triangle are congruent to the corresponding two angles and the included side of another triangle, then the triangles are congruent."

2. Write congruence statements for all corresponding parts and angle relationships of the pre-image and the image.

ACTIVITY
2.5

Non-Examples of Congruence Theorems

Thus far, you have explored and proven each of the triangle congruence theorems:

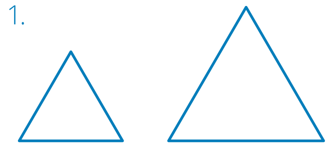
- Side-Side-Side Congruence Theorem (SSS)
- Side-Angle-Side Congruence Theorem (SAS)
- Angle-Side-Angle Congruence Theorem (ASA)

1. Juno wondered why AAA isn't on the list of congruence theorems. Provide a counterexample to show why Angle-Angle-Angle (AAA) *is not* considered a triangle congruence theorem.

2. Juno also wondered why SSA isn't on the list of congruence theorems. Provide a counterexample to show Juno why Side-Side-Angle (SSA) *is not* considered a triangle congruence theorem.

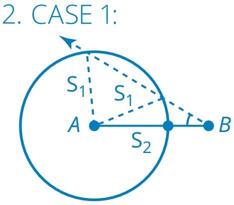
Answers

1.

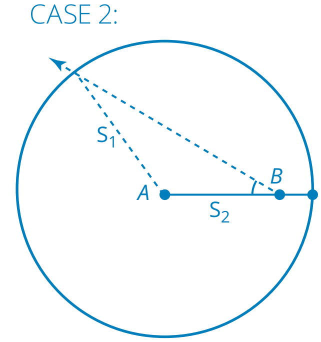


The measure of each interior angle of any equilateral triangle is 60°. Shown are two different equilateral triangles. All six angles are equal in measure, but the triangles are not congruent because the pairs of corresponding sides are not congruent. The pairs of corresponding sides are in fact proportional, but not congruent.

2. CASE 1:



CASE 2:



Consider line segment *AB* as the second 'S' in SSA. Point *A* is used as the center of a circle. The length of the circle radius is the first 'S' in SSA. The length of the radius of circle *A* is either longer (Case 2) or shorter (Case 1) than line segment *AB*. Both cases are drawn. In Case 2, if a ray is drawn at point *B* forming the given angle, or the 'A' in SSA, only one triangle is possible when the third vertex of the triangle is located on the circle. In Case 1, if a ray is drawn at point *B* forming the given angle or the 'A' in SSA, there are two possible locations for the third vertex of the triangle, both on circle *A*. Both triangles have SSA, but they are not congruent.

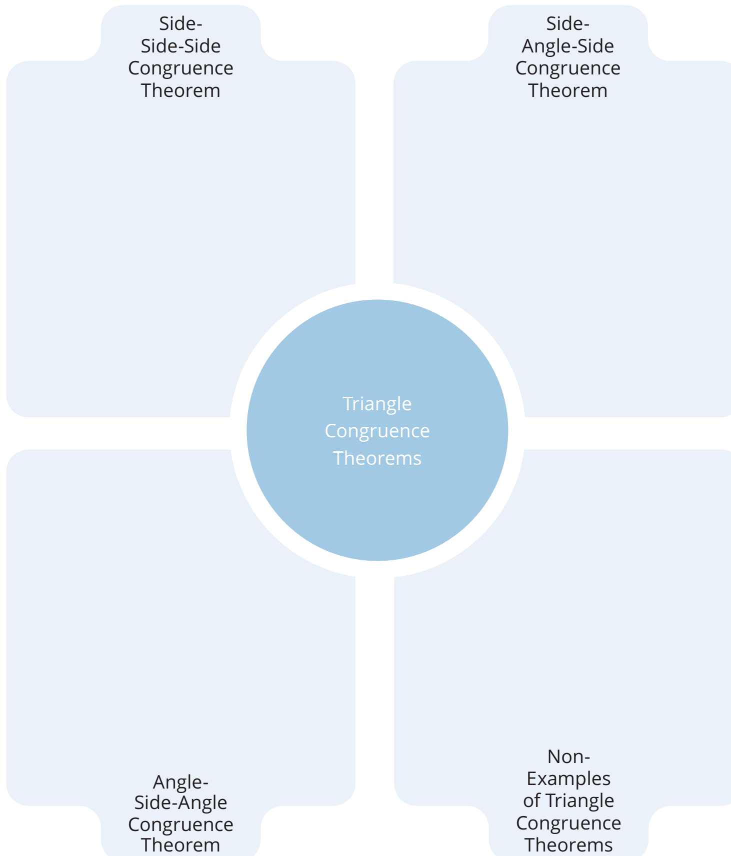
Answer

1. The graphic organizer entries should contain multiple representations such as diagrams, written explanations, and congruence statements.

TALK the TALK

The Right Combination

1. Complete the graphic organizer to summarize what you have learned about the triangle congruence theorems.



12 • TOPIC 3: Congruence Through Transformations