Congruence Through Transformations Summary

KEY TERMS

- counterexample
- truth value
- truth table
- spherical geometry
- Euclidean geometry
- Linear Pair Postulate
- Segment Addition Postulate
- Angle Addition Postulate
- Side-Side-Side Congruence Theorem (SSS)
- corresponding parts of congruent triangles are congruent (CPCTC)
- Side-Angle-Side Congruence Theorem (SAS)
- included angle
- Angle-Side-Angle Congruence Theorem (ASA)
- included side

LESSON 1 Elemental

When reasoning in mathematics, it is important to be able to identify false conclusions. There are two reasons why a conclusion may be false. Either the assumed information is false, or the conclusion does not necessarily follow from the hypothesis.

For example, consider the statement, “If you are reading this sentence, then your first language must be English.” This is an invalid argument. Both the hypothesis and conclusion might be true, but the conclusion does not necessarily follow from the hypothesis.

To show that a statement is false, you can provide a counterexample. A **counterexample** is a specific example that shows that a general statement is not true.

The **truth value** of a conditional statement is whether the statement is true or false. If a conditional statement could be true, then the truth value of the statement is considered true. The truth value of a conditional statement is either true or false, but not both.

You can identify the hypothesis and conclusion from a conditional statement. For example:

conditional statement

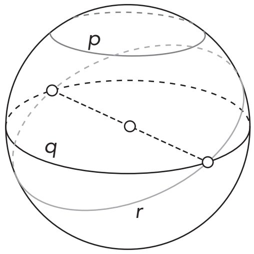
$$\underbrace{\text{If } x^2 = 36,}_{\text{Hypothesis}}\ \underbrace{\text{then } x = 6 \text{ or } x = -6}_{\text{Conclusion}}.$$

A **truth table** is a table that summarizes all possible truth values for a conditional statement $p \rightarrow q$. The first two columns of a truth table represent all possible truth values for the propositional variables p and q. The last column represents the truth value of the conditional statement $p \rightarrow q$.

p	q	$p \rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

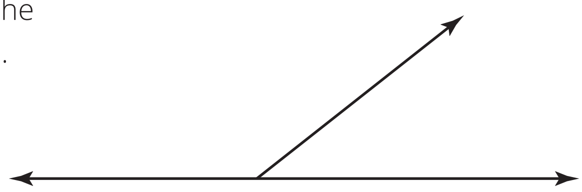
Euclid was a Greek mathematician who used a small number of undefined terms and postulates to systematically prove many theorems. As a result, Euclid was able to develop a complete system we now know as **Euclidean geometry**.

Spherical geometry, like its name implies, is a geometry that substitutes a sphere for a plane, which makes it different from plane geometry in significant ways. In spherical geometry, some topics that you have learned about, such as parallel lines and the sum of the interior angles of a triangle are very different.

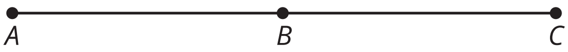


Three fundamental postulates—the Linear Pair Postulate, the Segment Addition Postulate, and the Angle Addition Postulate—can be used to make various conjectures. If the conjectures are proven, then they will become theorems.

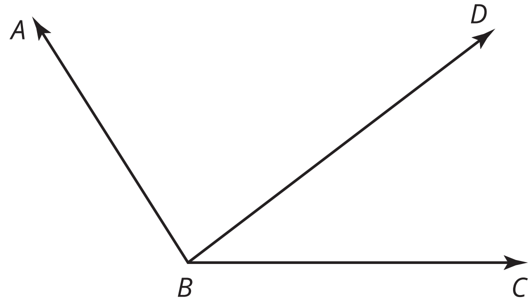
The **Linear Pair Postulate** states, "If two angles form a linear pair, then the angles are supplementary."



The **Segment Addition Postulate** states, "If point B is on $\overline{AC}$ and between points A and C, then $AB + BC = AC$."



The **Angle Addition Postulate** states, "If point D lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$."



LESSON
2

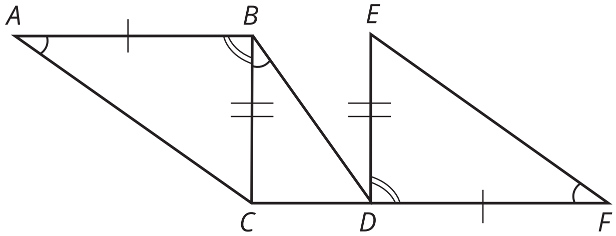
ASA, SAS, and SSS

It cannot be assumed that angles are right angles or that two figures are congruent, even if they appear to be. Labels and symbols indicate right angles and congruent figures.

Congruent line segments and congruent angles are often denoted using special markers rather than given measurements. Slash markers indicate congruent line segments. When multiple line segments contain a single slash marker, this implies that all of the line segments are congruent. Double and triple slash markers also denote other line segment congruencies. Arc markers indicate congruent angles. When multiple angles contain a single arc marker, this implies that those angles are congruent. Double and triple arc markers denote other angle congruencies.

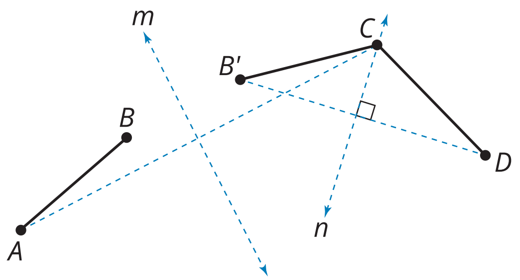
The markers on the diagram indicate congruent line segments and angles.

$\overline{AB} \cong \overline{DF}$ and $\overline{BC} \cong \overline{ED}$
$\angle BAC \cong \angle CBD \cong \angle DFE$
$\angle ABC \cong \angle FDE$



You can prove the conjecture that you can always map one congruent line segment onto the other using at most two reflections.

For example, suppose that $\overline{AB} \cong \overline{CD}$.



Since any point in the plane can be reflected across a line to map onto another point in the plane, point C is a reflection of point A across line m. Reflect $\overline{AB}$ across this line.

Reflections preserve distances, so $\overline{CB'} \cong \overline{CD}$, because $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{CB'}$. Thus, point C is equidistant from point B' and from point D.

Since a point is equidistant from two other points if and only if it lies on their perpendicular bisector, point C lies on the perpendicular bisector of $\overline{B'D}$ (line n). Thus, one last reflection across the perpendicular bisector maps $\overline{CB'}$ to $\overline{CD}$.

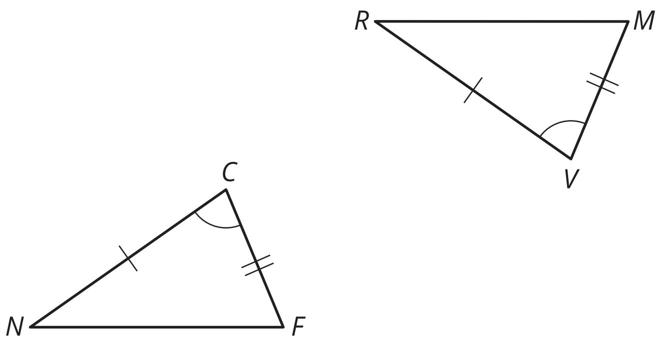
You can prove that triangles are congruent given different minimal criteria.

The **Side-Side-Side Congruence Theorem (SSS)** states, “If three sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent.”

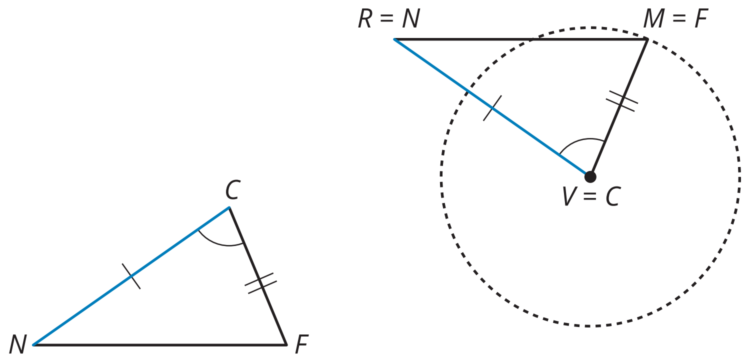
If two triangles are congruent, then each part of one triangle is congruent to the corresponding part of the other triangle. **Corresponding parts of congruent triangles are congruent**, abbreviated as **CPCTC**, is often used as a reason in proofs. CPCTC states that corresponding angles or sides in two congruent triangles are congruent. This reason can only be used after you have proven that the triangles are congruent.

The **Side-Angle-Side Congruence Theorem (SAS)** states, “If two sides and the included angle of one triangle are congruent to the corresponding two sides and the included angle of a second triangle, then the triangles are congruent.” An **included angle** is the angle formed by two sides of a triangle. Consider this proof of the Side-Angle-Side Theorem:

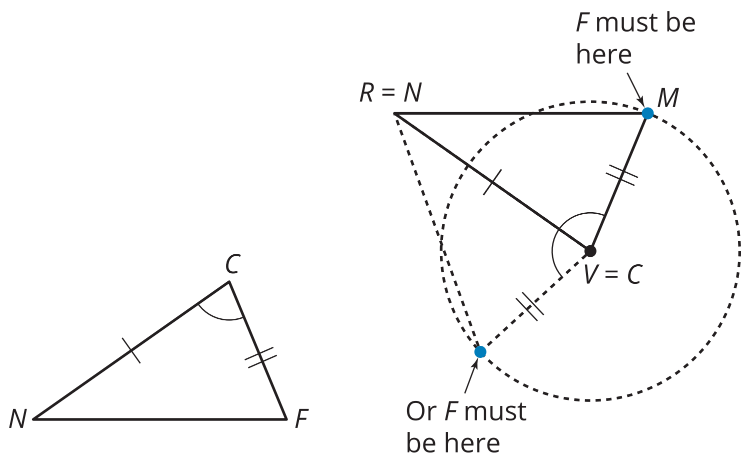
Given $\triangle CNF$ and $\triangle VRM$, $\overline{CN} \cong \overline{VR}$, $\overline{CF} \cong \overline{VM}$, and $\angle C \cong \angle V$.
You can map $\overline{CN}$ to $\overline{VR}$ in one or two reflections.



Since reflections preserve distance, this means that point F must be on the circle centered at V with radius CM, because $\overline{CF} \cong \overline{CM}$.



Since reflections preserve angle measures, there are only two possible locations for point F to be on the circle. If point F is not at M, then a reflection across $\overline{VR}$ will map point F to point M.



The **Angle-Side-Angle Congruence Theorem (ASA)** states, "If two angles and the included side of one triangle are congruent to the corresponding two angles and the included side of another triangle, then the triangles are congruent." An **included side** is a line segment between two consecutive angles of a figure. The proof of this theorem is similar to how the Side-Angle-Side Theorem is proven.

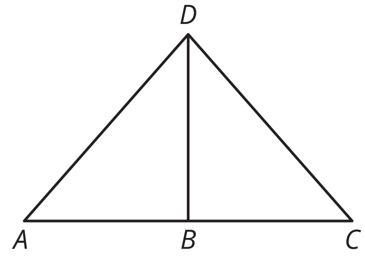
AAA and SSA are not triangle congruence theorems. Having these congruent parts of triangles does not mean that the triangles themselves are congruent.

LESSON 3

I Never Forget a Face

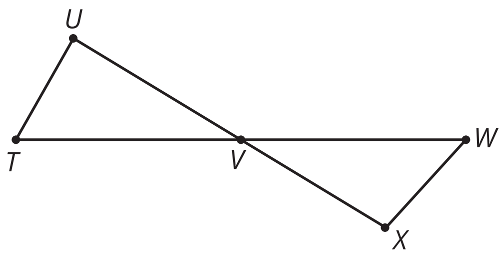
Given $\overline{AD} \cong \overline{DC}$ and $\overline{DB}$ bisects $\overline{AC}$.
You can use the Side-Side-Side Congruence Theorem to demonstrate that $\triangle ADB$ is congruent to $\triangle CDB$.

- Using the definition of a bisector, B is the midpoint of $\overline{AC}$ and therefore $\overline{AB} \cong \overline{BC}$.
- Since $\overline{DB}$ is the same side in each triangle, then $\overline{DB} \cong \overline{DB}$.
- Therefore, $\triangle ADB \cong \triangle CDB$ by SSS.



Given V is the midpoint of $\overline{UX}$ and V is the midpoint of $\overline{TW}$.
You can use the Side-Angle-Side Congruence Theorem to demonstrate that $\triangle UVT$ is congruent to $\triangle XVW$.

- By definition of a midpoint $\overline{UV} \cong \overline{XV}$ and $\overline{TV} \cong \overline{WV}$.
- Using the definition of vertical angles, $\angle UVT \cong \angle XVW$.
- Therefore, $\triangle UVT \cong \triangle XVW$ by SAS.



Given $\overline{WZ} \parallel \overline{XY}$ and $\overline{WX} \parallel \overline{ZY}$.
You can use the Angle-Side-Angle Congruence Theorem to demonstrate that $\triangle WXY$ is congruent to $\triangle YZW$.

- Using the definition of alternate interior angles, $\angle ZWY \cong \angle XYW$ and $\angle XWY \cong \angle ZYW$.
- Since $\overline{WY}$ is the same side in each triangle, then $\overline{WY} \cong \overline{WY}$.
- Therefore, $\triangle WXY \cong \triangle YZW$ by ASA.